# HYDRODYNAMICS AND VECTOR FIELD THEORY:

## Examples in Elementary Methods

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# EDITORS' FOREWORD

The object of this series is to provide texts in applied mathematics which will cover school and university requirements and extend to the post-graduate field. The texts will be convenient for use by pure and applied mathematicians and also by scientists and engineers wishing to acquire mathematical techniques to improve their knowledge of their own subjects. Applied mathematics is expanding at a great rate, and every year mathematical techniques are being applied in new fields of physical, biological and economic sciences. The series will aim at keeping abreast of these developments. Wherever new applications of mathematics arise it is hoped to persuade a leader in the field to describe them briefly.

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# PREFACE

The aim of this book is to exemplify the methods available for the solution of standard electrical, magnetic, and hydrodynamic problems. These methods are most clearly demonstrated through a wide selection of worked examples, such as is provided here. We hope that study of these will help to clarify both the techniques for dealing with problems of this sort and the physical consequences of the solutions obtained.

We have deferred to a second volume consideration of the more specialised mathematical techniques, e.g. complex variable, spherical harmonics, etc.

In each chapter the essential theory for a topic is quoted in note form, and is followed by a set of fully annotated solutions to specimen questions and by a further set of problems for solution by the student. Each chapter is, as far as possible, self-contained. All the questions used are taken from degree examinations set by British Universities during the past few years. They thus provide a representative selection of the present content of degree courses on these topics.

We are indebted to the authorities of the following Universities for permission to use questions from their papers.

BIRMINGHAM (B.)
CAMBRIDGE (C.)
DURHAM (D.)
EXETER (E.)
HULL (H.)
LEEDS (Le.)
LIVERPOOL (Li.)
LONDON (L.)
MANCHESTER (M.)
NOTTINGHAM (N.)
OXFORD (O.)
READING (R.)
SHEFFIELD (S.)
SOUTHAMPTON (So.)

The source of each problem is indicated in accordance with the key above. The responsibility for the solutions given here is, of course, ours alone. Proofs of bookwork are not, in general, given.

We are indebted to Mr. J. G. Henderson and Mr. Stephen Greener for the diagrams.

We shall be grateful for information about errors and suggestions for improvements.

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# CHAPTER 1

# VECTOR THEORY

In this chapter the problems deal purely with vector relationships and are not directly concerned with physical applications.

## 1.1 Notation and Basic Definitions

**1.1.1** The rules of vector algebra are assumed known and are not included here.*

**1.1.2** The notation used will be:

for *vector product* $\mathbf{a} \times \mathbf{b}$ (alternatives are $\mathbf{a} \wedge \mathbf{b}$ or $[\mathbf{ab}]$)
for *scalar product* $\mathbf{a} \cdot \mathbf{b}$ (alternative $(\mathbf{ab})$)
for *scalar triple product* $[\mathbf{abc}]$ (alternative $(\mathbf{a}, \mathbf{b}, \mathbf{c})$).

**1.1.3** The positive right-hand screw convention is used, in which a rotation and direction are connected in the same way as the rotation and direction of advance of a right-handed screw.

**1.1.4** The main functions associated with vector fields are defined in the table below.

(*a*) With reference to a volume $v$ bounded by a surface $S$ whose unit *outward* normal vector is $\mathbf{n}$. (*N.B.* In some texts the inward normal vector is used, which alters the sign of the corresponding term.)

(*b*) With reference to orthogonal curvilinear coordinates $x_1$, $x_2$, $x_3$ whose corresponding elements of length are $h_1 dx_1$, $h_2 dx_2$, $h_3 dx_3$.

(*c*) With reference to Cartesian coordinates $x$, $y$, $z$.

The functions are all invariant, i.e. their values do not depend on the particular coordinate system chosen. $\phi$ is a scalar and $\mathbf{q}$ $(q_1, q_2, q_3)$ a vector quantity.

**1.1.5** TABLE OF VALUES OF VECTOR FIELD FUNCTIONS

| | (*a*) | (*b*) | (*c*) |
|---|---|---|---|
| grad $\phi$ or $\nabla\phi$ | $\operatorname{Lim}_{v \to 0} \frac{1}{v}\int_S \mathbf{n}\phi \, dS$ | $\frac{1}{h_1}\frac{\partial\phi}{\partial x_1}, \frac{1}{h_2}\frac{\partial\phi}{\partial x_2}, \frac{1}{h_3}\frac{\partial\phi}{\partial x_3}$ | $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}$ |
| div $\mathbf{q}$ or $\nabla \cdot \mathbf{q}$ | $\operatorname{Lim}_{v \to 0} \frac{1}{v}\int_S \mathbf{n} \cdot \mathbf{q} \, dS$ | $\frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial x_1}(h_2 h_3 q_1) + \frac{\partial}{\partial x_2}(h_3 h_1 q_2) + \frac{\partial}{\partial x_3}(h_1 h_2 q_3)\right\}$ | $\frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} + \frac{\partial q_3}{\partial z}$ |

* They may be found in any standard text, e.g. Weatherburn, *Vector Analysis*, Bell and Sons.

TABLE OF VALUES OF VECTOR FIELD FUNCTIONS—*Continued*

| | (*a*) | (*b*) | (*c*) |
|---|---|---|---|
| curl $\mathbf{q}$ or $\nabla \times \mathbf{q}$ | $\lim_{v \to 0} \frac{1}{v}\int_S \mathbf{n} \times \mathbf{q} dS$ | 1st component: $\frac{1}{h_2 h_3}\left\{\frac{\partial}{\partial x_2}(h_3 q_3) - \frac{\partial}{\partial x_3}(h_2 q_2)\right\}$ and other components by cyclic permutation | $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ q_1 & q_2 & q_3 \end{vmatrix}$ $\mathbf{i}, \mathbf{j}, \mathbf{k}$ being unit vectors in $x, y, z$ directions |
| div grad $\phi$ or $\nabla^2\phi$ | | $\frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial x_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \phi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \phi}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \phi}{\partial x_3}\right)\right\}$ | $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$ |
| $(\mathbf{a} \,.\, \text{grad})\,\mathbf{q}$ or $(\mathbf{a} \,.\, \nabla)\mathbf{q}$ | | 1st component: $\frac{a_1}{h_1}\frac{\partial q_1}{\partial x_1} + \frac{a_2}{h_2}\frac{\partial q_2}{\partial x_2} + \frac{a_3}{h_3}\frac{\partial q_1}{\partial x_3}$ and other components by cyclic permutation | 1st component: $a_1\frac{\partial q_1}{\partial x} + a_2\frac{\partial q_1}{\partial y} + a_3\frac{\partial q_1}{\partial z}$ and other components by cyclic permutation |

## 1.2 Relations between Vector Field Functions

$$
\begin{aligned}
\text{curl grad } \phi &= 0, \text{ div curl } \mathbf{q} = 0 \text{ identically} \\
\text{curl curl } \mathbf{q} &= \text{grad div } \mathbf{q} - \boldsymbol{\nabla}^2\mathbf{q} \\
\text{div } (\phi\mathbf{q}) &= \text{grad } \phi \,.\, \mathbf{q} + \phi \text{ div } \mathbf{q} \\
\text{curl } (\phi\mathbf{q}) &= \text{grad } \phi \times \mathbf{q} + \phi \text{ curl } \mathbf{q} \\
\text{div } (\mathbf{a} \times \mathbf{b}) &= \mathbf{b} \,.\, \text{curl } \mathbf{a} - \mathbf{a} \,.\, \text{curl } \mathbf{b} \\
\text{curl } (\mathbf{a} \times \mathbf{b}) &= (\mathbf{b} \,.\, \nabla)\mathbf{a} - (\mathbf{a} \,.\, \nabla)\mathbf{b} + \mathbf{a} \text{ div } \mathbf{b} - \mathbf{b} \text{ div } \mathbf{a} \\
\text{grad } (\mathbf{a} \,.\, \mathbf{b}) &= (\mathbf{b} \,.\, \nabla)\,\mathbf{a} + (\mathbf{a} \,.\, \nabla)\mathbf{b} + \mathbf{b} \times \text{curl } \mathbf{a} + \mathbf{a} \times \text{curl } \mathbf{b}
\end{aligned}
$$

The vector operator $\boldsymbol{\nabla}^2$ must be distinguished from the scalar $\nabla^2$, meaning divgrad.

In Cartesian coordinates *only*, $\boldsymbol{\nabla}^2\mathbf{q} = (\nabla^2 q_1, \nabla^2 q_2, \nabla^2 q_3.)$

In any other coordinate system, $\nabla^2\mathbf{q} = \text{grad div } \mathbf{q} - \text{curl curl } \mathbf{q}$. (See **7.1**)

## 1.3 Integral Theorems

### (a) Green's or Gauss's theorem

When $S$ is a simple closed surface, $\mathbf{n}$ the unit outward normal vector, enclosing a volume $v$, and $\phi$, $\mathbf{q}$ and their derivatives are continuous at all points of $v$, then we have the different forms

$$\text{(i)} \int_v \text{grad } \phi \, dv = \int_v \nabla \, \phi \, dv = \int_S \mathbf{n} \, \phi \, dS$$

(ii) $\int_v \operatorname{div} \mathbf{q}\, dv = \int_v \nabla . \mathbf{q}\, dv = \int_S \mathbf{n} . \mathbf{q}\, dS$

(iii) $\int_v \operatorname{curl} \mathbf{q}\, dv = \int_v \nabla \times \mathbf{q}\, dv = \int_S \mathbf{n} \times \mathbf{q}\, dS$

A particular application of (ii) is

(iv) $\int_S \mathbf{n} . \operatorname{grad} \left(\frac{1}{r}\right) dS = 4\pi$ or 0 according as the origin, i.e. the point from which $r$ is measured, is inside or outside $S$.

(*N.B.* The expression $\mathbf{n}\, dS$ is sometimes written as $d\mathbf{S}$, which is then taken to mean a vector of magnitude $dS$ in the direction of $\mathbf{n}$).

### (b) Stokes' theorem

When $C$ is a simple closed curve, $S$ a surface bounded by $C$, $\mathbf{n}$ its unit normal, positive (right-hand screw) relative to the sense of description of $C$, $d\mathbf{s}$ a vector, magnitude $ds$, where $ds$ is an element of arc of $C$, and $\mathbf{q}$ and its derivatives are continuous on $S$, then

(i) $\oint_C \mathbf{q} . d\mathbf{s} = \int_S \operatorname{curl} \mathbf{q} . \mathbf{n}\, dS$

An application of this is that

(ii) curl $\mathbf{q} = 0$ everywhere is the necessary and sufficient condition that there exists a scalar $\phi$ such that

$$\mathbf{q} = -\operatorname{grad} \phi$$

and that $\oint \mathbf{q} . d\mathbf{s} = 0$ for all closed curves and $\int_A^B \mathbf{q} . d\mathbf{s}$ is independent of the path $AB$ and depends only on the positions of $A$ and $B$.

### (c) Green's second theorem

If $U$, $V$ are scalar functions, one-valued and continuous, with their derivatives, in a volume $v$ bounded by a simple closed surface $S$, then

$$\text{(i)} \int_v (\operatorname{grad} U . \operatorname{grad} V)\, dv = -\int_v U\nabla^2 V\, dv + \int_S U \frac{\partial V}{\partial n}\, dS$$

$$= -\int_v V\nabla^2 U\, dv + \int_S V \frac{\partial U}{\partial n}\, dS$$

and hence

$$\text{(ii)} \int_v (U\nabla^2 V - V\nabla^2 U)\, dv = \int_S \left(U \frac{\partial V}{\partial n} - V \frac{\partial U}{\partial n}\right) dS$$

## Solved Problems

### Problem 1

(*a*) Show that $[(\mathbf{a} \times \mathbf{b}), (\mathbf{a} \times \mathbf{c}), \mathbf{d}] = (\mathbf{a} \cdot \mathbf{d})\, [\mathbf{a}, \mathbf{b}, \mathbf{c}]$

(*b*) If $(\mathbf{c} - \mathbf{a}) \cdot \mathbf{a} = (\mathbf{c} - \mathbf{b}) \cdot \mathbf{b}$ prove $\mathbf{a} - \mathbf{b}$ perpendicular to $\mathbf{c} - \mathbf{a} - \mathbf{b}$. (E.)

---

**Solution.** (*a*) In a triple scalar product the cross may be put in either place and so the left-hand side may be written

$$(\mathbf{a} \times \mathbf{b}) \cdot \{(\mathbf{a} \times \mathbf{c}) \times \mathbf{d}\} = (\mathbf{a} \times \mathbf{b})\{(\mathbf{d} \cdot \mathbf{a})\mathbf{c} - (\mathbf{d} \cdot \mathbf{c})\mathbf{a}\}$$

(by expansion of the triple vector product)

$$= [\mathbf{abc}](\mathbf{a} \cdot \mathbf{d})$$

since $\mathbf{a} \times \mathbf{b} \cdot \mathbf{a} = 0$ and $\mathbf{d} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{d}$.

(*b*) If two vectors are perpendicular their scalar product is zero. Hence we consider

$$(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{c} - \mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot (\mathbf{c} - \mathbf{a}) - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot (\mathbf{c} - \mathbf{b}) + \mathbf{b} \cdot \mathbf{a} = 0$$

Hence the two vectors are perpendicular.

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**Comment.** These are straightforward applications of vector algebra.

---

### Problem 2

**1.** Establish the relations

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = -\, [\mathbf{a}, \mathbf{c}, \mathbf{b}]$$
$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

for the scalar and vector triple products, and

**2.** Prove that

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}]\mathbf{d} = [\mathbf{d}, \mathbf{b}, \mathbf{c}]\mathbf{a} + [\mathbf{a}, \mathbf{d}, \mathbf{c}]\mathbf{b} + [\mathbf{a}, \mathbf{b}, \mathbf{d}]\mathbf{c}.$$

**3.** If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are linearly independent prove that

$$\mathbf{A} = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]},\ \mathbf{B} = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]},\ \mathbf{C} = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

are also linearly independent, and that

$$\mathbf{a} = \frac{\mathbf{B} \times \mathbf{C}}{[\mathbf{A}, \mathbf{B}, \mathbf{C}]},\ \mathbf{b} = \frac{\mathbf{C} \times \mathbf{A}}{[\mathbf{A}, \mathbf{B}, \mathbf{C}]},\ \mathbf{c} = \frac{\mathbf{A} \times \mathbf{B}}{[\mathbf{A}, \mathbf{B}, \mathbf{C}]}$$

**4.** Hence prove that an arbitrary vector $\mathbf{d}$ may be written

$$\mathbf{d} = (\mathbf{A} \cdot \mathbf{d})\mathbf{a} + (\mathbf{B} \cdot \mathbf{d})\mathbf{b} + (\mathbf{C} \cdot \mathbf{d})\mathbf{c}$$
$$= (\mathbf{a} \cdot \mathbf{d})\mathbf{A} + (\mathbf{b} \cdot \mathbf{d})\mathbf{B} + (\mathbf{c} \cdot \mathbf{d})\mathbf{C} \qquad \text{(H.)}$$

---

**Solution.**

**1.** The relations for the scalar and vector triple products are bookwork.

**2.** To prove this assume

(*a*) **a**, **b**, **c** not coplanar. Then there exist unique scalars $X$, $Y$, $Z$ such that

$$\mathbf{d} = X\mathbf{a} + Y\mathbf{b} + Z\mathbf{c} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Operating on both sides of (1) with . $\mathbf{b} \times \mathbf{c}$ gives

$[\mathbf{d}, \mathbf{b}, \mathbf{c}] = X[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ with the other terms vanishing. Hence

$$X = \frac{[\mathbf{d}, \mathbf{b}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} \text{ and } Y \text{ and } Z \text{ are found similarly.}$$

(*b*) If **a**, **b**, **c** are coplanar, then $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0$.

If **d** is also coplanar all the triple products vanish.

If **d** is not coplanar with **a**, **b**, **c** then, putting $\mathbf{c} = X\mathbf{a} + Y\mathbf{b}$, the right-hand side of the required relationship becomes

$$X[\mathbf{d}, \mathbf{b}, \mathbf{a}]\mathbf{a} + Y[\mathbf{a}, \mathbf{d}, \mathbf{b}]\mathbf{b} + [\mathbf{a}, \mathbf{b}, \mathbf{d}](X\mathbf{a} + Y\mathbf{b})$$

and since $$[\mathbf{d}, \mathbf{b}, \mathbf{a}] = [\mathbf{a}, \mathbf{d}, \mathbf{b}] = -[\mathbf{a}, \mathbf{b}, \mathbf{d}]$$

this is zero and the result is proved.

**3.** The necessary and sufficient condition for three vectors **a**, **b**, **c** to be linearly independent is $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \neq 0$.

We form the triple scalar product for **A**, **B**, **C**.

Writing $D$ for $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ we have

$$\mathbf{B} \times \mathbf{C} = \frac{(\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b})}{D^2} = \frac{\mathbf{a}}{D} \text{ (after simplification)} \quad . \quad (2)$$

Hence $$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \frac{\mathbf{b} \times \mathbf{c} \cdot \mathbf{a}}{D^2} = \frac{1}{D} \neq 0 \quad . \quad . \quad . \quad (3)$$

Hence **A**, **B**, **C** are also linearly independent.

$$\text{From (2), } \mathbf{a} = D(\mathbf{B} \times \mathbf{C}) = \frac{\mathbf{B} \times \mathbf{C}}{[\mathbf{A}, \mathbf{B}, \mathbf{C}]} \text{ from (3)} \quad . \quad (4)$$

**4.** Rearranging the expressions obtained in (2), any vector **d** may be written

$$\mathbf{d} = \frac{\mathbf{b} \times \mathbf{c} \cdot \mathbf{d}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}\mathbf{a} + \frac{\mathbf{c} \times \mathbf{a} \cdot \mathbf{d}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}\mathbf{b} + \frac{\mathbf{a} \times \mathbf{b} \cdot \mathbf{d}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}\mathbf{c}$$
$$= (\mathbf{A} \cdot \mathbf{d})\mathbf{a} + (\mathbf{B} \cdot \mathbf{d})\mathbf{b} + (\mathbf{C} \cdot \mathbf{d})\mathbf{c}$$

and, similarly, putting **d** in terms of the linearly independent **A**, **B**, **C** and using the reciprocal relation (4),

$$\mathbf{d} = (\mathbf{a} \cdot \mathbf{d})\mathbf{A} + (\mathbf{b} \cdot \mathbf{d})\mathbf{B} + (\mathbf{c} \cdot \mathbf{d})\mathbf{C}$$

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**Comment.** The relations arise directly from the definition of linear independence.

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**Problem 3**

(i) Solve for $\mathbf{x}$ the vector equation

$$K\mathbf{x} + \mathbf{a} \times \mathbf{x} = \mathbf{c}$$

in which $K$ is a scalar not zero.

(ii) Obtain the general solution of the vector equation

$$K\mathbf{x} + (\mathbf{a} \cdot \mathbf{x})\mathbf{b} = \mathbf{c}$$

Investigate the nature of the solution when the general solution is inapplicable owing to $K$ having a special value. (E.)

---

**Solution.** (i)
$$K\mathbf{x} + \mathbf{a} \times \mathbf{x} = \mathbf{c} \quad \dots \dots \quad (1)$$

Multiply both sides vectorially by $\mathbf{a}$ in order to break up the vector product, which gives

$$K\mathbf{x} \times \mathbf{a} + a^2\mathbf{x} - (\mathbf{a} \cdot \mathbf{x})\mathbf{a} = \mathbf{c} \times \mathbf{a} \quad \dots \quad (2)$$

Similarly multiply both sides of (1) scalarly by $\mathbf{a}$, giving

$$K\mathbf{x} \cdot \mathbf{a} = \mathbf{c} \cdot \mathbf{a} \quad \dots \dots \dots \quad (3)$$

Substituting in (2) for $\mathbf{a} \cdot \mathbf{x}$ gives

$$K\mathbf{x} \times \mathbf{a} + a^2\mathbf{x} - \frac{(\mathbf{c} \cdot \mathbf{a})}{K}\mathbf{a} = \mathbf{c} \times \mathbf{a} \quad \dots \dots \quad (4)$$

Eliminating $\mathbf{a} \times \mathbf{x}$ from (1) and (4) gives

$$\mathbf{x} = \frac{K^2\mathbf{a} + (\mathbf{c} \cdot \mathbf{a})\mathbf{a} + K(\mathbf{c} \times \mathbf{a})}{K(K^2 + a^2)}$$

and this is a genuine solution, by substitution; it is also the only solution.

(ii)
$$K\mathbf{x} + (\mathbf{a} \cdot \mathbf{x})\mathbf{b} = \mathbf{c} \quad \dots \dots \quad (1)$$

Since a scalar product cannot be broken up, it must be substituted for, and so we multiply scalarly by $\mathbf{a}$, giving

$$(\mathbf{a} \cdot \mathbf{x})(K + \mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{c}$$

i.e.
$$\mathbf{a} \cdot \mathbf{x} = \frac{\mathbf{a} \cdot \mathbf{c}}{K + (\mathbf{a} \cdot \mathbf{b})} \text{ if } \mathbf{a} \cdot \mathbf{b} \neq -K \quad \dots \dots \quad (2)$$

Substitution in (1) gives

$$\mathbf{x} = \frac{1}{K}\mathbf{c} - \frac{1}{K}\left\{\frac{\mathbf{a} \cdot \mathbf{c}}{K + (\mathbf{a} \cdot \mathbf{b})}\right\}\mathbf{b} \text{ if } K \neq 0$$

---

This solution breaks down when $K = -\mathbf{a}\,.\,\mathbf{b}$ and when $K = 0$.

(*a*) When $K = -\mathbf{a}\,.\,\mathbf{b}$ equation (1) is

$$-(\mathbf{a}\,.\,\mathbf{b})\mathbf{x} + (\mathbf{a}\,.\,\mathbf{x})\mathbf{b} = \mathbf{c}$$

i.e.
$$(\mathbf{x} \times \mathbf{b}) \times \mathbf{a} = \mathbf{c}$$

so that $\mathbf{a} \perp \mathbf{c}$ and $\mathbf{a}\,.\,\mathbf{c} = 0$.

Also $(\mathbf{x} \times \mathbf{b}) \perp \mathbf{c}$, i.e. $(\mathbf{x} \times \mathbf{b})\,.\,\mathbf{c} = 0$, so that $\mathbf{x}$ is coplanar with $\mathbf{b}$ and $\mathbf{c}$. Hence we may put

$$\mathbf{x} = \alpha\mathbf{b} + \beta\mathbf{c} \ (\alpha, \beta \text{ scalars}),$$

giving, as $\mathbf{a}\,.\,\mathbf{c} = 0$,

$$-(\mathbf{a}\,.\,\mathbf{b})\alpha\mathbf{b} - (\mathbf{a}\,.\,\mathbf{b})\beta\mathbf{c} + (\mathbf{a}\,.\,\mathbf{b})\alpha\mathbf{b} = \mathbf{c}$$

Thus $\alpha$ is arbitrary, $\beta = -\dfrac{1}{\mathbf{a}\,.\,\mathbf{b}} = \dfrac{1}{K}$ and the general solution in this case is

$$\mathbf{x} = \frac{\mathbf{c}}{K} + \alpha\mathbf{b}$$

which includes the form previously obtained.

---

(*b*) When $K = 0$, $(\mathbf{a}\,.\,\mathbf{x})\mathbf{b} = \mathbf{c}$, which implies

$$\mathbf{b} \,//\, \mathbf{c}, \text{ i.e. } \mathbf{b} = \lambda\mathbf{c}$$

Then $(\mathbf{a}\,.\,\mathbf{x}) = 1/\lambda$ and $\mathbf{x}$ is any vector whose component along $\mathbf{a}$ is known, i.e
$$\mathbf{x} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma(\mathbf{a} \times \mathbf{b})$$
where
$$\mathbf{a}\,.\,\mathbf{x} = 1/\lambda = \alpha\mathbf{a}^2 + \beta\mathbf{a}\,.\,\mathbf{b}$$

Thus $\beta = \dfrac{(1/\lambda) - \alpha\mathbf{a}^2}{\mathbf{a}\,.\,\mathbf{b}}$, $\gamma$ is arbitrary and the general solution is

$$\mathbf{x} = \alpha\mathbf{a} + \left(\frac{\dfrac{1}{\lambda} - \alpha\mathbf{a}^2}{\mathbf{a}\,.\,\mathbf{b}}\right)\mathbf{b} + \gamma(\mathbf{a} \times \mathbf{b})$$

---

**Problem 4**

The centroid of a triangle $ABC$ is $P$, sides $\overline{BC} = \mathbf{a}$ and $\overline{CA} = \mathbf{b}$ being infinitesimal. If $\phi_Q$ is the value of a finite single-valued differentiable function $\phi$ at the point $Q$, prove from first principles that

$$\int_{ABC} \phi d\mathbf{s} = \tfrac{1}{2}\{(\mathbf{a} \times \mathbf{b}) \times (\text{grad } \phi)_P\}$$

Hence or otherwise express $\oint \phi d\mathbf{s}$, $\oint \mathbf{q}\,.\,d\mathbf{s}$ and $\oint \mathbf{q} \times d\mathbf{s}$ round a closed curve $C$ as surface integrals over a surface $S$ bounded by $C$. (L.)

---

**Solution.**

Since $P$ is the centroid of the triangle, $\overline{BP}$ is the mean of vectors drawn from $B$ to the three corners, i.e. of $\mathbf{a} + \mathbf{b}$, 0 and $\mathbf{a}$, and hence

$$\overline{BP} = \tfrac{2}{3}\mathbf{a} + \tfrac{1}{3}\mathbf{b}$$

Similarly
$$\overline{AP} = -\tfrac{1}{3}\mathbf{a} - \tfrac{2}{3}\mathbf{b}$$
$$\overline{CP} = -\tfrac{1}{3}\mathbf{a} + \tfrac{1}{3}\mathbf{b}$$

At $X$ on $AB$, where $\overline{AX} = x(\mathbf{a} + \mathbf{b})$ $(x \leqslant 1)$, we have, as a first (linear) approximation,

$$\phi_X = x\phi_A + (1 - x)\phi_B,$$

and, relating all $\phi$ to $\phi_P$, we have also as a first approximation

$$\phi_A = \phi_P + \overline{PA} \,.\, (\text{grad}\,\phi)_P,$$

with similar expressions for $\phi_B$, $\phi_C$.

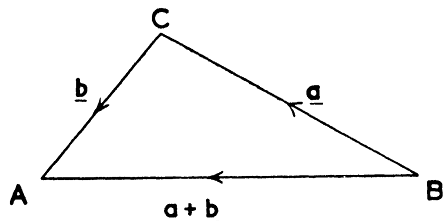


Fig. 1

Thus
$$\int_{AB} \phi d\mathbf{s} = -(\mathbf{a} + \mathbf{b}) \int_0^1 \{x\phi_A + (1 - x)\phi_B\} dx$$
$$= -(\mathbf{a} + \mathbf{b})\tfrac{1}{2}(\phi_B + \phi_A)$$
$$= +(\mathbf{a} + \mathbf{b})\{-\phi_P + \tfrac{1}{2}(\overline{BP} + \overline{AP}) \,.\, (\text{grad}\,\phi)_P\}$$

Similarly
$$\int_{BC} \phi d\mathbf{s} = \mathbf{a}\{\phi_P - \tfrac{1}{2}(\overline{CP} + \overline{BP}) \,.\, (\text{grad}\,\phi)_P\}$$
$$\int_{CA} \phi d\mathbf{s} = \mathbf{b}\{\phi_P - \tfrac{1}{2}(\overline{AP} + \overline{CP}) \,.\, (\text{grad}\,\phi)_P\}$$

Hence
$$\int_{ABC} \phi d\mathbf{s} = \tfrac{1}{2}\{(\text{grad}\,\phi)_P \,.\, (\overline{AP} - \overline{CP})\}\mathbf{a}$$
$$+ \tfrac{1}{2}\{(\text{grad}\,\phi)_P \,.\, (\overline{BP} - \overline{CP})\}\mathbf{b}$$
$$= -\tfrac{1}{2}\{(\text{grad}\,\phi)_P \,.\, \mathbf{b}\}\mathbf{a} + \tfrac{1}{2}\{(\text{grad}\,\phi)_P \,.\, \mathbf{a}\}\mathbf{b}$$
$$= \tfrac{1}{2}\{(\mathbf{a} \times \mathbf{b}) \times (\text{grad}\,\phi)_P\}$$

by the expansion rule for the vector triple product.

Now $\tfrac{1}{2}(\mathbf{a} \times \mathbf{b})$ is the vector $\mathbf{n}\, dS$, where $dS$ is the magnitude of the area of triangle $ABC$ and $\mathbf{n}$ is the unit vector normal to the plane of

$ABC$ and in the positive sense relative to the sense of description of $ABC$. Hence this result means that for an *elemental* triangle

$$\int \phi \, d\mathbf{s} = \mathbf{n} \times \operatorname{grad} \phi \, dS$$

Now any surface $S$ bounded by a closed curve $C$ may be split up into a large number of such triangles: when each of these triangles is described in the same sense the total effect will be to describe each side twice, once in each direction, with the exception only of the sides which make up $C$, the boundary curve, which are described once only. The total effect of covering the perimeter of all the elemental triangles is thus to cover the bounding curve, $C$, once.

Thus
$$\sum \int_{ABC} \phi \, d\mathbf{s} = \int_C \phi \, d\mathbf{s}$$

and we have
$$\int_{ABC} \phi \, d\mathbf{s} = \mathbf{n} \times \operatorname{grad} \phi \, dS, \text{ whence}$$

$$\sum \int_{ABC} \phi \, d\mathbf{s} = \int_S \mathbf{n} \times \operatorname{grad} \phi \, dS$$

$\mathbf{n}$ is the unit outward normal to each elemental triangle, positive w.r.t. the sense $ABC$, and so, in the limit, it becomes the unit outward normal to the surface $S$, positive w.r.t. the sense of description of the curve $C$.

Hence
$$\oint_C \phi \, d\mathbf{s} = \int_S \mathbf{n} \times \operatorname{grad} \phi \, dS$$

---

Applying this to the three components of a vector $\mathbf{q}$,

$$\oint_C \mathbf{q} \cdot d\mathbf{s} = \int_S \mathbf{n} \cdot \operatorname{curl} \mathbf{q} \, dS$$

---

either by expansion, or by writing $\oint \mathbf{q} \cdot d\mathbf{s} = \int (\mathbf{n} \times \nabla) \cdot \mathbf{q} \, dS$ and converting the right-hand side symbolically to $\int \mathbf{n} \cdot \nabla \times \mathbf{q} \, dS$, the required form.

Similarly, $\oint_C \mathbf{q} \times d\mathbf{s} = \int_S \{\mathbf{n} \operatorname{div} \mathbf{q} - (\mathbf{n} \cdot \nabla)\mathbf{q} + \operatorname{curl} \mathbf{q} \times \mathbf{n}\} \, dS$, most easily proved by expansion.

---

**Comment.** This problem is solved by the technique used in proving many of the basic results—namely the splitting up of a region into elemental zones, in each of which certain simple relations can be shown to hold.

**Problem 5**

(*a*) Prove that the position vector of the centroid of a volume $v$ bounded by a closed surface $S$ is $\frac{1}{2}v\int \mathbf{n}\, r^2\, dS$, and hence find the centroid of a hemisphere.

(*b*) If $OA$, $OB$, $OC$ are three edges of a rectangular box, of lengths $a, b, c$, calculate $\int \mathbf{r}\,.\,d\mathbf{S}$ over the six faces of the box, $\mathbf{r}$ being the position vector with origin $O$. Apply Gauss's theorem to verify the result. (L.)

---

**Solution.** (*a*) The centroid of a volume has position vector $\bar{\mathbf{r}}$, where

$$\bar{\mathbf{r}} = \frac{1}{v}\int_v \mathbf{r}\, dv$$

But
$$\mathbf{r} = \tfrac{1}{2}\,\text{grad}\,(r^2),\ \text{and so}$$

$$\bar{\mathbf{r}} = \frac{1}{2v}\int_v \text{grad}\,(r^2)\, dv$$

$$= \frac{1}{2v}\int_S \mathbf{n}\, r^2\, dS \text{ by } \mathbf{(1.3)}\ (a)\ \text{(i)}$$

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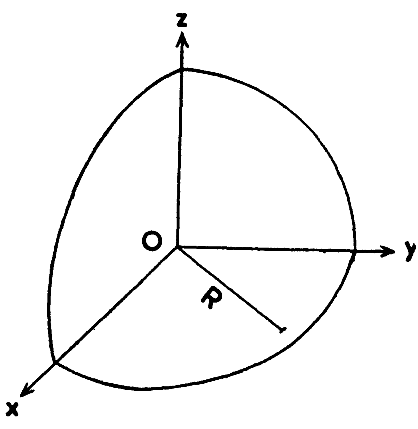


Fig. 2

Applying this to a hemisphere, centre of base $O$, radius $a$, base perpendicular to $Oz$, the element of area of the curved surface, $dS_1$, say, is $2\pi a\, dz$ and of the base $dS_2 = 2\pi R\, dR$.
The unit outward normal, $\mathbf{n}$, w.r.t. axes $Ox$, $Oy$, $Oz$, is

for $S_1$
$$\left(\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right) = \mathbf{n}_1$$

for $S_2$
$$(0, 0, -1) = \mathbf{n}_2$$

and $r^2$, the square of the position vector, is $a^2$ for the curved area, $R^2$ for the base. Hence $\bar{\mathbf{r}} = \dfrac{1}{\frac{4}{3}\pi a^3}\left\{\int_{S_1} a^2\mathbf{n}_1\, 2\pi a\, dz + \int_{S_2} R^2\mathbf{n}_2\, 2\pi R\, dR\right\}$

Now $\int_{S_1} x\, dz = \int_{S_1} y\, dz = 0$ by symmetry, and so $\bar{\mathbf{r}}$ has a component along $Oz$ only (as is obvious), and this component is

$$\bar{r}_z = \frac{1}{\frac{4}{3}\pi a^3}\left\{\int_0^a 2\pi a^3 \frac{z}{a}\, dz - \int_0^a 2\pi R^3\, dR\right\}$$
$$= \frac{1}{\frac{4}{3}\pi a^3}(\pi a^2 \,.\, a^2 - \tfrac{1}{2}\pi a^4) = \tfrac{3}{8}a$$

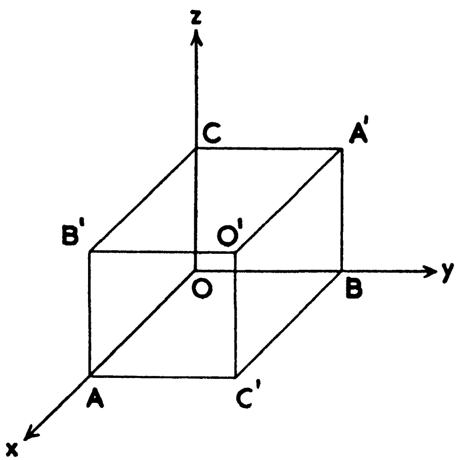


Fig. 3

(*b*) Over faces $OBCA^1$, $OCAB^1$, $OABC^1$

$$\int \mathbf{r} \,.\, d\mathbf{S} = 0$$

since $\mathbf{r}$ is in the face and $d\mathbf{S}$ perpendicular to it.

Over face $OB^1C^1A$, $d\mathbf{S}$ is $(1, 0, 0)\, dS$ everywhere.

Hence $\quad \int \mathbf{r} \,.\, d\mathbf{S} = \int x\, dS$ and $x = a$, $\int dS = bc$

so $\quad \int \mathbf{r} \,.\, d\mathbf{S} = abc$

By symmetry this is also true over the other two faces meeting at $O^1$; hence, over the whole cube,

$$\int \mathbf{r} \,.\, d\mathbf{S} = 3abc$$

By Gauss's theorem **1.3 (a) (ii)** $\int \mathbf{r} \, . \, d\mathbf{S} = \int \text{div}\, \mathbf{r}\, dv$ and by differentiation, as $r^2 = x^2 + y^2 + z^2$, div $\mathbf{r} = 3$.

Thus $\int \text{div}\, \mathbf{r}\, dv = 3\int dv = 3abc$ as before.

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**Problem 6**

Show that if div $\mathbf{A} = 0$, $\nabla^2\mathbf{A} = \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{A}}{\partial t^2}$ and

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \ \mathbf{H} = \text{curl}\, \mathbf{A}, \text{ then}$$

**1.** $\text{curl}\, \mathbf{H} - \dfrac{1}{c}\dfrac{\partial \mathbf{E}}{\partial t} = 0$

**2.** $\text{curl}\, \mathbf{E} + \dfrac{1}{c}\dfrac{\partial \mathbf{H}}{\partial t} = 0$

**3.** $\text{div}\, \mathbf{H} = 0 = \text{div}\, \mathbf{E}$

**4.** Show that $\mathbf{A} = \mathbf{i}a_1 \cos\left\{\dfrac{2\pi}{\lambda}(z - ct)\right\} + \mathbf{j}a_2 \sin\left\{\dfrac{2\pi}{\lambda}(z - ct)\right\}$

is a solution, where $a_1$, $a_2$, $\lambda$ are constants and $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$. Obtain **E** and **H** and show that they rotate about the $z$-axis with frequency $c/\lambda$. (H.)

---

**Solution.**

**1.**
$$\text{curl}\, \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \text{curl curl}\, \mathbf{A} + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}$$
$$= \text{grad div}\, \mathbf{A} - \nabla^2\, \mathbf{A} + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} \ \text{(by } \mathbf{1.2}\text{)}$$
$= 0$ by the relations given for **A**

**2.**
$$\text{curl}\, \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = \text{curl}\left(-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) + \frac{1}{c}\frac{\partial}{\partial t}(\text{curl}\, \mathbf{A})$$
$= 0$ since the order of $\partial/\partial t$ and curl may be interchanged, one being a time and one a space variation.

**3.** div $\mathbf{H}$ = div curl $\mathbf{A} = 0$ identically **(1.2)**

$$\text{div}\, \mathbf{E} = \text{div}\left(-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) = 0, \text{ interchanging order of div and } \frac{\partial}{\partial t}$$

**4.** When $\mathbf{A} = \mathbf{i}a_1 \cos \alpha + \mathbf{j}\, a_2 \sin \alpha$ where $\alpha = \dfrac{2\pi}{\lambda}(z - ct)$

$$\text{div}\, \mathbf{A} = \frac{\partial}{\partial x}(a_1 \cos \alpha) + \frac{\partial}{\partial y}(a_2 \sin \alpha) = 0$$

$$\text{and } \nabla^2 \mathbf{A} = \mathbf{i}\left\{\frac{\partial^2}{\partial x^2}(a_1 \cos\alpha) + \frac{\partial^2}{\partial y^2}(a_1 \cos\alpha) + \frac{\partial^2}{\partial z^2}(a_1 \cos\alpha)\right\}$$

$$+ \mathbf{j}\left\{\frac{\partial^2}{\partial x^2}(a_2 \sin\alpha) + \frac{\partial^2}{\partial y^2}(a_2 \sin\alpha) + \frac{\partial^2}{\partial z^2}(a_2 \sin\alpha)\right\}$$

(from (**1.2**)

$$= -\mathbf{i}\, a_1 \frac{4\pi^2}{\lambda^2}\cos\alpha - \mathbf{j}\, a_2 \frac{4\pi^2}{\lambda^2}\sin\alpha \left(\text{since } \frac{\partial\alpha}{\partial z} = \frac{2\pi}{\lambda}\right)$$

$$\text{and } \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{1}{c^2}\mathbf{i}\frac{\partial^2}{\partial t^2}(a_1\cos\alpha) + \frac{1}{c^2}\mathbf{j}\frac{\partial^2}{\partial t^2}(a_2\sin\alpha)$$

$$= -\mathbf{i}\, a_1 \frac{4\pi^2}{\lambda^2}\cos\alpha - \mathbf{j}\, a_2 \frac{4\pi^2}{\lambda^2}\sin\alpha \left(\text{since } \frac{\partial\alpha}{\partial t} = -\frac{2\pi c}{\lambda}\right)$$

$$\text{Thus} \qquad \nabla^2 \mathbf{A} = \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}$$

Hence, with this **A**, all the required relations are satisfied.

$$\text{In this case, } \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\mathbf{i}\frac{2\pi}{\lambda}a_1 \sin\alpha + \mathbf{j}\, a_2\frac{2\pi}{\lambda}\cos\alpha$$

$$\mathbf{H} = \operatorname{curl} \mathbf{A} = -\mathbf{i}\frac{\partial}{\partial z}(a_2 \sin\alpha) + \mathbf{j}\frac{\partial}{\partial z}(a_1\cos\alpha)$$

$$= -\mathbf{i}\frac{2\pi}{\lambda}a_2\cos\alpha - \mathbf{j}\frac{2\pi}{\lambda}a_1\sin\alpha$$

Thus both **E** and **H** are vectors perpendicular to $OZ$.

For fixed $z$, **E** and **H** change with time. Since both depend on $\alpha = \frac{2\pi}{\lambda}(z - ct)$, a change in $t$ by any multiple of $\frac{\lambda}{c}$ will restore the original value, and thus **E** and **H** rotate with period $\frac{\lambda}{c}$, i.e. frequency $\frac{c}{\lambda}$, about the axis of $z$.

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**Comment.** These equations occur in electromagnetic theory. Their proof depends wholly on the definitions and relations in **1.1** and **1.2**.

**Problem 7**

**1.** Prove that div $\mathbf{B} = 0$ is the necessary and sufficient condition that a given vector **B** can be expressed in the form $\mathbf{B} = \operatorname{curl} \mathbf{A}$.

**2.** If $\mathbf{B} = \operatorname{grad}\phi$ where $\phi = \mathbf{M}\,.\operatorname{grad}\left(\frac{1}{r}\right)$ and **M** is a constant vector, verify that $\mathbf{A} = \operatorname{grad}\left(\frac{1}{r}\right) \times \mathbf{M}$ is a solution, satisfying also div $\mathbf{A} = 0$.

**3.** Find **A** with $A_3 = 0$ if $\mathbf{B} = \left(\frac{-yz}{r^5}, \frac{-xz}{r^5}, \frac{2xy}{r^5}\right)$ (L.)

---

**Solution.** **1.** (*a*) If $\mathbf{B} = \text{curl } \mathbf{A}$, then $\text{div } \mathbf{B} = \text{div curl } \mathbf{A} = 0$ identically, and so $\text{div } \mathbf{B} = 0$ is a necessary condition.

(*b*) If div **B**, $\dfrac{\partial B_1}{\partial x} + \dfrac{\partial B_2}{\partial y} + \dfrac{\partial B_3}{\partial z} = 0$

we define three components $A_1$, $A_2$, $A_3$ by the equations

$$\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = B_1 \qquad \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} = B_2$$

This may always be done, e.g. by taking $A_3 = 0$,

$$\frac{\partial A_2}{\partial z} = -B_1, \ \frac{\partial A_1}{\partial z} = B_2$$

so that
$$A_1 = \int B_2 \, dz + f_1(x, y)$$
$$A_2 = \int -B_1 \, dz + f_2(x, y)$$

where $f_1, f_2$ are arbitrary functions.

Then $\dfrac{\partial A_2}{\partial x} - \dfrac{\partial A_1}{\partial y} = \displaystyle\int -\frac{\partial B_1}{\partial x} dz - \int \frac{\partial B_2}{\partial y} dz + \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}$

$$= \int \frac{\partial B_3}{\partial z} dz + \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \text{ (since div } \mathbf{B} = 0)$$

$= B_3$, since $f_1$ and $f_2$, being arbitrary, can be chosen to satisfy $\dfrac{\partial f_2}{\partial x} = \dfrac{\partial f_1}{\partial y}$. Hence, given $\text{div } \mathbf{B} = 0$, values of **A** defined in this way will satisfy $\mathbf{B} = \text{curl } \mathbf{A}$ (in fact, there is an infinite number of such vectors **A**), and so the condition is sufficient.

**2.** If $\mathbf{B} = \text{grad } \phi$, where $\phi = \mathbf{M} \,.\, \text{grad}\left(\dfrac{1}{r}\right)$, **M** a constant, then $\text{div } \mathbf{B} = \nabla^2\phi$.

$$\text{grad}\left(\frac{1}{r}\right) \text{ is } \left\{\frac{\partial}{\partial x}\left(\frac{1}{r}\right), \frac{\partial}{\partial y}\left(\frac{1}{r}\right), \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right\} = \left(-\frac{x}{r^3}, -\frac{y}{r^3}, -\frac{z}{r^3}\right)$$

so that $\phi = -\dfrac{M_1 x}{r^3} - \dfrac{M_2 y}{r^3} - \dfrac{M_3 z}{r^3}$, and each of these terms separately satisfies $\nabla^2\phi = 0$ (as can be proved by differentiation).

Hence $\text{div } \mathbf{B} = 0$ as required.

Consider $\mathbf{A} = \text{grad}\left(\dfrac{1}{r}\right) \times \mathbf{M}$

Then $\text{curl } \mathbf{A} = \text{grad}\left(\dfrac{1}{r}\right) \text{div } \mathbf{M} - \left\{\text{grad}\left(\dfrac{1}{r}\right) . \nabla\right\}\mathbf{M}$

$$-M\nabla^2\left(\frac{1}{r}\right) + (\mathbf{M} \,.\, \nabla) \text{ grad}\left(\frac{1}{r}\right) \text{ (from 1.2)}$$

$$= (\mathbf{M} \,.\, \nabla) \text{ grad}\left(\frac{1}{r}\right), \text{ since } \mathbf{M} \text{ is constant and } \nabla^2\left(\frac{1}{r}\right) = 0$$

Also

$$\begin{aligned}\mathbf{B} &= \text{grad}\left\{\mathbf{M}\,.\,\text{grad}\left(\frac{1}{r}\right)\right\}\\ &= (\mathbf{M}\,.\,\nabla)\,\text{grad}\left(\frac{1}{r}\right) + \left\{\text{grad}\left(\frac{1}{r}\right).\,\nabla\right\}\mathbf{M}\\ &\quad + \mathbf{M}\times\text{curl grad}\left(\frac{1}{r}\right) + \text{grad}\left(\frac{1}{r}\right)\times\text{curl}\,\mathbf{M}\ \text{(from 1.2)}\\ &= (\mathbf{M}\,.\,\nabla)\,\text{grad}\left(\frac{1}{r}\right)\ (\mathbf{M}\ \text{constant and curl grad} = 0)\\ &= \text{curl}\,\mathbf{A}.\end{aligned}$$

Further

$$\text{div}\,\mathbf{A} = -\text{grad}\left(\frac{1}{r}\right).\,\text{curl}\,\mathbf{M} + \mathbf{M}\,.\,\text{curl grad}\left(\frac{1}{r}\right) = 0.$$

Thus these values of **A** and **B** satisfy the given relations.

3. Given $\mathbf{B} = \left(-\frac{yz}{r^5}, -\frac{xz}{r^5}, \frac{2xy}{r^5}\right)$ and $A_3 = 0$ we have, as in part 1(*b*),

$$A_1 = \int B_2\,dz + f_1(x, y) = \frac{1}{3}\frac{x}{r^3} + f_1(x, y)$$

$$A_2 = \int -B_1\,dz + f_2(x, y) = -\frac{1}{3}\frac{y}{r^3} + f_2(x, y)$$

and we require $$\frac{\partial f_2}{\partial x} = \frac{\partial f_2}{\partial y}$$

It may be verified that these satisfy $\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} = \frac{2xy}{r^5}$.

---

**Problem 8**

Using Green's theorem in the form

$$\int\left(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n}\right)dS = \int_v (U\nabla^2 V - V\nabla^2 U)\,dv$$

show that, if $P$ is any point of the volume $v$ bounded by surface $S$,

$$4\pi V_P = \int_S\left\{\frac{1}{R}\frac{\partial V}{\partial n} - V\frac{\partial}{\partial n}\left(\frac{1}{R}\right)\right\}dS - \int_v \frac{1}{R}\nabla^2 V\,dv$$

where $V$ is a scalar function of position satisfying conditions that should be stated, $V_P$, is the value of $V$ at $P$ and $R$ is the distance from $P$ to a variable point of $v$.

Hence:

(i) Show that $$4\pi = -\int_S \frac{\partial}{\partial n}\left(\frac{1}{R}\right)dS$$

(ii) Determine $V$ as a function of position inside a sphere of unit radius, given that $\nabla^2 V = -6$ everywhere inside the sphere and that everywhere on the boundary $V = 4$, $\frac{\partial V}{\partial n} = -2$ (L.)

---

**Solution.** Applying Green's theorem to the volume $v^1$ bounded externally by $S$ and internally by $S^1$, a small sphere, centre $P$, radius $\varepsilon$ (so that $P$ is not a point of $v^1$), we have, if $V$ be a finite, single-valued function which, with its derivatives, is continuous in $v^1$, and, if $U = \frac{1}{R}$,

$$\int_{S+S^1}\left\{\frac{1}{R}\frac{\partial V}{\partial n} - V\frac{\partial}{\partial n}\left(\frac{1}{R}\right)\right\} dS = \int_{v^1}\left\{\frac{1}{R}\nabla^2 V - V\nabla^2\left(\frac{1}{R}\right)\right\} dv$$

$$= \int_{v^1}\frac{1}{R}\nabla^2 V\, dv, \text{ as } \nabla^2\left(\frac{1}{R}\right) = 0 \text{ in } v.$$

But $$\int_{S^1}\left\{\frac{1}{R}\frac{\partial V}{\partial n} - V\frac{\partial}{\partial n}\left(\frac{1}{R}\right)\right\} dS = \int_{S^1}\frac{1}{\varepsilon}\frac{\partial V}{\partial n}\, dS - \int_{S^1} V\,.\,\frac{1}{\varepsilon^2}\, dS$$

(since the outward drawn normal to $S^1$ is directed towards $P$)

Also $$\int_{S^1}\frac{\partial V}{\partial n}\, dS = \int_{v_1}\operatorname{div} V\, dv \simeq \tfrac{4}{3}\pi\varepsilon^3\,(\operatorname{div} V)_P$$

$v_1$ being the small sphere; hence

$$\frac{1}{\varepsilon}\int_{S^1}\frac{\partial V}{\partial n}\, dS \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0 \text{ and}$$

$$\int_{S^1} V\,.\,\frac{1}{\varepsilon^2}\, dS \longrightarrow V_P\,.\,\frac{1}{\varepsilon^2}\,4\pi\varepsilon^2 \longrightarrow 4\pi V_P \text{ as } \varepsilon \longrightarrow 0$$

Also $$\int_{v^1}\frac{1}{R}\nabla^2 V\, dv \longrightarrow \int_{v}\frac{1}{R}\nabla^2 V\, dv, \text{ since } dv \text{ is } 0\,(R^3) \text{ near } P$$

Hence $$4\pi V_P = -\int_{S^1}\left\{\frac{1}{R}\frac{\partial V}{\partial n} - V\frac{\partial}{\partial n}\left(\frac{1}{R}\right)\right\} dS$$

$$= -\int_{v}\frac{1}{R}\nabla^2 V\, dv + \int_{S}\left\{\frac{1}{R}\frac{\partial V}{\partial n} - V\frac{\partial}{\partial n}\left(\frac{1}{R}\right)\right\} dS$$

---

(i) Putting $V = 1$ everywhere,

$$4\pi = -\int_{S}\frac{\partial}{\partial n}\left(\frac{1}{R}\right) dS, \text{ since the other terms vanish.}$$

(ii)

Let $P$ be distant $s$ from $O$, the centre of the sphere, take $PO$ as the base-line, $O$ the origin for polar coordinates $r$, $\theta$.

Then $$R^2 = r^2 + s^2 - 2rs\cos\theta$$

and for points on the boundary

$$R^2 = 1 + s^2 - 2s\cos\theta$$

Since $\nabla^2 V = -6$ inside the sphere, and $V = 4$,

$$\frac{\partial V}{\partial n} = -2 \text{ on the boundary,}$$

$$4\pi V_P = +6\int \frac{1}{R}\, dv - 2\int_S \frac{1}{R}\, dS - 4\int_S \frac{\partial}{\partial n}\left(\frac{1}{R}\right) dS$$

Evaluating the integrals in turn, first

$$\int \frac{1}{R}\, dv = \int_0^1 r^2\, dr \int_0^\pi \frac{2\pi \sin\theta\, d\theta}{(r^2 + s^2 - 2rs\cos\theta)^{1/2}}$$

$$= \int_0^1 \frac{2\pi r}{s}\left[(r^2 + s^2 + 2rs)^{1/2} - (r^2 + s^2 - 2rs)^{1/2}\right]$$

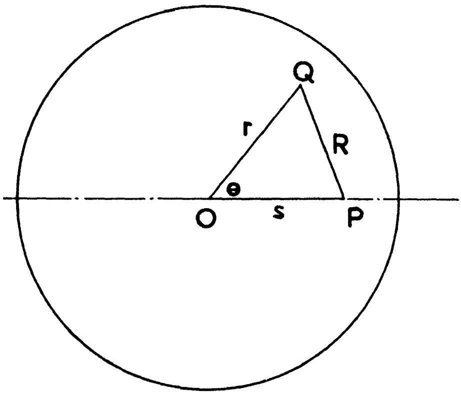


Fig. 4

The square roots must be positive as they are the values of $R$ at $\theta = 0$ and $\theta = \pi$. Hence for $r<s$, the second is $s - r$, and for $r>s$ it becomes $r - s$.

Thus the integral is $\displaystyle\int_0^s \frac{2\pi r}{s} \,.\, (2r)\, dr + \int_s^1 \frac{2\pi r}{s}\, (2s)\, dr$

$$= 2\pi\left(1 - \tfrac{1}{3}s^2\right)$$

Secondly, $\displaystyle\int \frac{1}{R}\, dS = \int_0^\pi \frac{2\pi \sin\theta\, d\theta}{(1 + s^2 - 2s\cos\theta)^{1/2}} = \frac{2\pi}{s}(2s) = 4\pi$

and thirdly, $\displaystyle\int \frac{\partial}{\partial n}\left(\frac{1}{R}\right) dS = -4\pi$ from (i)

Hence $\qquad 4\pi V_s = 6 \,.\, 2\pi(1 - \tfrac{1}{3}s^2) - 2(4\pi) - 4(-4\pi)$

and so $\qquad V_s = 3 - s^2 + 2 = 5 - s^2$

Now if any other line be taken as base line, so that $P$ becomes the point $(r, \theta)$, the value of $V$ depends only on the distance $OP$, i.e. at $(r, \theta)$ the value of $V$ is $5 - r^2$.

*Alternatively*, it is shorter to derive $V$ not using the integral relations. Since $\nabla^2 V$, and $V$, $\dfrac{\partial V}{\partial n}$, on the boundary, are all constant, $V$ must be a function of $r$ only.

Then the equations to be satisfied are

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = -6$$

($\nabla^2$ in spherical polar coordinates)

$$\frac{dV}{dr} = -2,\ r = 1$$

$$V = \quad 4,\ r = 1$$

where the partial derivatives may be regarded as ordinary derivatives, since $V$ depends only on $r$.

Integrating the first equation gives

$$r^2\frac{dV}{dr} = -2r^3 + C_1$$

whence

$$\frac{dV}{dr} = -2r + C_1 r^{-2}$$

and, therefore,

$$V = -r^2 - C_1 r^{-1} + C_2$$

When $r = 1$, $V = 4$ and $\dfrac{dV}{dr} = -2$, giving

$$C_1 = 0,\ C_2 = 5$$

and hence, as before,

$$V = 5 - r^2$$

---

**Problem 9**

Prove Gauss's integral theorem, and, assuming that $\phi$, $\psi$ are twice differentiable functions of position in a volume $V$ and on its bounding surface $S$, deduce Green's theorem.

$$\int(\phi\nabla^2\psi - \psi\nabla^2\phi)\,dv = \int(\phi \text{ grad } \psi - \psi \text{ grad } \phi)\,dS$$

A solution of the wave equation $\dfrac{\partial^2\psi}{\partial t^2} - c^2\nabla^2\psi = 0$

is of the form $\psi = \phi(r)f(t)$. Prove that if $f(t)$ is periodic, then $\phi$ satisfies $\nabla^2\phi + K^2\phi = 0$, where $K$ is a constant.

Show that $\{\exp(iKr)\}/r$ is a solution of this equation in any region not

including the origin, and use Green's theorem to prove that, for every solution, the value of $\phi$ at the origin is given by

$$-\frac{1}{4\pi}\iint_S\left\{\phi\frac{\partial}{\partial n}\left(\frac{e^{iKr}}{r}\right)-\frac{e^{iKr}}{r}\frac{\partial\phi}{\partial n}\right\}dS$$

assuming that the origin is within $S$. (L.)

---

**Solution.** The first part is bookwork.

In the second part assume $\psi = \phi(r)f(t)$ and $f(t) = e^{i\omega t}$, a general periodic function of period $\frac{2\pi}{\omega}$.

The wave equation then reduces to

$$-\omega^2\phi(r)e^{i\omega t} - c^2\nabla^2\phi e^{i\omega t} = 0$$

i.e.
$$\nabla^2\phi + \frac{\omega^2}{c^2}\phi = 0$$

i.e.
$$\nabla^2\phi + K^2\phi = 0$$

where $K^2 = \frac{\omega^2}{c^2}$, a constant.

---

Using the expression for $\nabla^2\phi$, with $\phi$ a function of $r$ only, gives

$$\frac{1}{r^2}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + K^2\phi = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Substituting $\phi = e^{iKr}/r$, we find

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = \frac{1}{r^2}\frac{d}{dr}(-e^{iKr} + riKe^{iKr})$$

$$= \frac{1}{r^2}(-iKe^{iKr} + iKe^{iKr} - K^2re^{iKr})$$

$$= -\frac{K^2}{r}e^{iKr}$$

Thus $\phi = e^{iKr}/r$ is a solution of (1); but it is not continuous at the origin.

Let $\phi'$ be another solution of (1) and use Green's theorem in a region $v^1$, excluding the origin, between a small sphere $S^1$, centre the origin and within $S$, and the surface $S$.

Then $$\int_{v^1}(\phi'\nabla^2\phi - \phi\nabla^2\phi')\,dv = \int_{S+S^1}\left\{\phi'\frac{\partial}{\partial n}\left(\frac{e^{iKr}}{r}\right)-\frac{e^{iKr}}{r}\frac{\partial\phi'}{\partial n}\right\}dS$$

and $\nabla^2\phi = -K^2\phi$, $\nabla^2\phi' = -K^2\phi'$, whence the left-hand side is zero. On the right-hand side

$$\int_{S^1} \phi' \frac{\partial}{\partial n}\left(\frac{e^{iKr}}{r}\right) dS = -\int_{S^1} \phi' \frac{\partial}{\partial r}\left(\frac{e^{iKr}}{r}\right) dS$$

$$= -\int_{S^1} (\phi')_0 \left\{-\frac{e^{iK\varepsilon}}{\varepsilon^2} + iK\frac{e^{iK\varepsilon}}{\varepsilon}\right\} dS$$

$$\longrightarrow 4\pi\phi'_0 \text{ as } \varepsilon \longrightarrow 0$$

$\varepsilon$ being the radius of $S^1$. Also

$$\int_{S^1} \frac{e^{iKr}}{r}\frac{\partial\phi'}{\partial n} dS = \frac{e^{iK\varepsilon}}{\varepsilon}\int_{S^1}\frac{\partial\phi'}{\partial n} dS = \frac{e^{iK\varepsilon}}{\varepsilon}\int_{v^1} \operatorname{div} \phi' \, dv$$

(where $v_1$ is the volume inside $S^1$)

$$= \frac{e^{iK\varepsilon}}{\varepsilon} \cdot \frac{4}{3}\pi\varepsilon^3 (\operatorname{div} \phi')_0 \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0$$

Thus
$$\phi'_0 = \frac{1}{4\pi}\int_S\left\{\phi'\frac{\partial}{\partial n}\left(\frac{e^{iKr}}{r}\right) - \frac{e^{iKr}}{r}\frac{\partial\phi'}{\partial n}\right\} dS$$

---

**Comment.** This is another application of the method used in Problem 7 to avoid a singularity.

---

**Problem 10**

A scalar function $V$ is such that grad $V = \mathbf{F}$ is parallel to $(3x - y + 2z, -2x - 2z, x - y)$ at every point except points in the plane $x = y$. Determine $V$, and prove also that the $\mathbf{F}$-lines are plane curves. (H.)

---

**Solution.** It is given that

$$\frac{\partial V}{\partial x} : \frac{\partial V}{\partial y} : \frac{\partial V}{\partial z} = 3x - y + 2z : -2x - 2z : x - y \quad . \quad (1)$$

but the coefficient of proportionality differs from point to point.

However, it follows identically that

$$\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} - \frac{\partial V}{\partial z} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = -\frac{dz}{1} = \frac{dV}{0}$$

of which particular integrals are $V = a$, $x - y = b$, $x + z = c$, where $a$, $b$, $c$ are constants. Thus the general solution of (2) is

$$V = f\{(x - y), (x + z)\}$$

Put $x - y = u$, $x + z = v$ so that $\dfrac{\partial V}{\partial u} = -\dfrac{\partial V}{\partial y}$ and

$$\frac{\partial V}{\partial v} = \frac{\partial V}{\partial z}. \text{ Also } \frac{\partial V}{\partial y} : \frac{\partial V}{\partial z} = -\frac{\partial V}{\partial u} : \frac{\partial V}{\partial v} = -2v : u,$$

$$\text{i.e. } u\frac{\partial V}{\partial u} - 2v\frac{\partial V}{\partial v} = 0$$

The auxiliary equations are

$$\frac{du}{u} = \frac{dv}{-2v} = \frac{dV}{0}$$

of which particular integrals are $V = a$, $uv^{1/2} = b$, and so the general solution is

$$V = f(uv^{1/2})$$
$$= f\{(x - y)(x + z)^{1/2}\}$$

where $f$ is an arbitrary function. This is the most general form for $V$ which satisfies the given relation.

An **F**-line is a curve whose tangent at any point is in the direction of **F**; thus its direction cosines satisfy $l + m - n = 0$, and so the line is everywhere perpendicular to the vector $(1, 1, -1)$ and the whole curves lie in planes perpendicular to this vector.

---

**Comment.** This method reduces the solution of a vector problem to consideration of the allied differential equations (Lagrange's linear equation).

## PROBLEMS FOR SOLUTION

1. If the scalar triple product $\mathbf{p} \times \mathbf{q} . \mathbf{r}$ is denoted by $[\mathbf{p}, \mathbf{q}, \mathbf{r}]$ show that

(i) $(\mathbf{b} \times \mathbf{c}) . (\mathbf{d} \times \mathbf{a}) + (\mathbf{c} \times \mathbf{a}) . (\mathbf{d} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{b}) . (\mathbf{d} \times \mathbf{c}) = 0$

(ii) $[(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}, (\mathbf{b} \times \mathbf{c}) \times \mathbf{a}, (\mathbf{c} \times \mathbf{a}) \times \mathbf{b}] = 0$

(iii) $(\mathbf{b} \times \mathbf{c}) \times (\mathbf{d} \times \mathbf{a}) + (\mathbf{c} \times \mathbf{a}) \times (\mathbf{d} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{b}) \times (\mathbf{d} \times \mathbf{c})$
$= 2[\mathbf{a}, \mathbf{b}, \mathbf{c}]\mathbf{d}$ (H.)

2. (i) If $A$, $B$, $C$, and $D$ have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ and numbers, $\lambda$, $\mu$, $\nu$, not all zero, exist such that

$$(\lambda + \mu + \nu)\mathbf{d} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$$

show that $A$, $B$, $C$, and $D$ are coplanar. What geometrical property corresponds to $\lambda = 0$?

(ii) Show that, if two pairs of opposite edges of a tetrahedron are perpendicular, the third pair are also perpendicular. (D.)

3. Find the general solution of the vector equation for $\mathbf{x}$

$$\mathbf{x} \times \mathbf{a} = \mathbf{b} \text{ given that } \mathbf{a} . \mathbf{b} = 0$$

Hence or otherwise obtain the general solution for $\mathbf{x}$ and $\mathbf{y}$ of the simultaneous equations

$$\mathbf{x} + \mathbf{y} = \mathbf{a}$$
$$\mathbf{x} \times \mathbf{y} = \mathbf{b} \text{ given that } \mathbf{a} \cdot \mathbf{b} = 0$$

What is the particular solution which satisfies the further condition

$$\mathbf{x} \cdot \mathbf{a} = \mathbf{a}^2?$$ (E.)

4. Prove the formulae

$$\begin{aligned}\operatorname{div}(\phi\mathbf{A}) &= \phi \operatorname{div} \mathbf{A} + \mathbf{A} \cdot \operatorname{grad} \phi\\ \operatorname{curl}(\phi\mathbf{A}) &= \phi \operatorname{curl} \mathbf{A} - \mathbf{A} \times \operatorname{grad} \phi\\ \operatorname{curl}(\mathbf{A} \times \mathbf{B}) &= \mathbf{A} \operatorname{div} \mathbf{B} - \mathbf{B} \operatorname{div} \mathbf{A} + \mathbf{B}_i \frac{\partial \mathbf{A}}{\partial x_i} - \mathbf{A}_i \frac{\partial \mathbf{B}}{\partial x_i}\end{aligned}$$

If $\mathbf{r}$ is the position vector of a point in space evaluate

$$\operatorname{div}(r^n\mathbf{r}), \operatorname{curl}(r^n\mathbf{r})$$

If $\mathbf{a}$ is a constant vector show that curl $(\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}$. (D.)

5. (i) Prove div $(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \operatorname{curl} \mathbf{A} - \mathbf{A} \cdot \operatorname{curl} \mathbf{B}$.

(ii) If $\mathbf{B}\phi = \operatorname{grad} \psi$, where $\mathbf{B}$ is a vector function of position and $\phi$, $\psi$ are scalar functions of position, show that $\mathbf{B} \cdot \operatorname{curl} \mathbf{B} = 0$.

(iii) A vector function of position $\mathbf{A}$ is such that the line integral

$$\int_0^P \mathbf{A} \cdot d\mathbf{s} = K^2 \operatorname{div} \mathbf{A}$$

where $K^2$ is a constant and div $\mathbf{A}$ is evaluated at the point $P$. Show that

$$\mathbf{A} = K^2\nabla^2\mathbf{A}$$

6. Establish Green's theorem in the form

$$\int_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) dS = \int_v (\phi\nabla^2\psi - \psi\nabla^2\phi)dv$$

A vector field $\mathbf{F}(\mathbf{r})$ is $0(|\mathbf{r}|^{-2})$ at infinity, and is such that curl $\mathbf{F} = 0$ everywhere. Show that at any general point $\mathbf{r}_0$

$$4\pi\mathbf{F}(\mathbf{r}_0) = -\nabla_0 \int \frac{(\nabla \cdot \mathbf{F})}{|\mathbf{r} - \mathbf{r}_0|} dv$$

where $\nabla$ and $\nabla_0$ denote the gradient at $\mathbf{r}$ and $\mathbf{r}_0$ respectively, and the integral is taken through all space. (C.)

7. A volume $v$ is bounded externally by a sphere of radius $a$, centre $C$, and internally by a sphere $S^1$, centre $C^1$, which lies wholly within $S$. State Green's theorem and deduce that

$$\int_S \left(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n}\right) dS + \int_S \left(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n}\right) dS = 0$$

where $U$, $V$ are functions of position such that $\nabla^2 U = \nabla^2 V = 0$ at all points of $v$ and $\frac{\partial}{\partial n}$ denotes differentiation along the normal drawn outwards from $v$.

A general point $P$ inside $S$ is at a distance $R$ from $C^1$ and at a distance $R^1$ from the inverse point of $C^1$ with regard to $S$; if

$$U = 1/R - a/bR^1$$

where $b = CC^1$, prove that $\nabla^2 U = 0$ at all points inside $S$ (except $C^1$) and that, if $P$ is on $S$, $U = 0$ and $\dfrac{\partial U}{\partial n} = -\dfrac{(a^2 - b^2)}{aR^3}$

Hence show that
$\frac{a^2 - b^2}{a}\int_S \frac{VdS}{R^3} + \int_{S^1}\left(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n}\right)dS = 0$ and derive the limiting form of this equation as the radius of $S^1$ tends to zero. (L.)

8. State the necessary and sufficient conditions that a vector may be represented as the gradient of a potential function. Hence prove that the integral over all space of the scalar product of an irrotational vector **A** and a solenoidal vector **B** is zero, provided **A** and **B** and their derivatives are continuous everywhere except on a finite number of closed surfaces across which the tangential components of **A** and the normal components of **B** are continuous. It is assumed that $|\mathbf{A}|$ and $|\mathbf{B}|$ are $0(1/r^2)$ for large values of $r$.

9. (i) Establish the identity curl grad $\phi = 0$ by applying Stokes' theorem to grad $\phi$.

(ii) Show that if the vector **q** may be expressed in the form $\phi_1\nabla\phi_2$, then **q** is perpendicular to curl **q**.

(iii) Express the vector $(xy^2z, x^2yz, x^2y^2)$ in the form $\phi_1\nabla\phi_2$. (M.)

10. State Stokes' theorem.

If **t** is a constant vector, prove that

$$\mathbf{n}\,.\,\text{curl}\,(\mathbf{F}\times\mathbf{t}) = \mathbf{t}\,.\,(\mathbf{n}\times\nabla)\times\mathbf{F}$$

Hence or otherwise deduce that if **F** is a vector field which together with its first space derivatives is finite, continuous, and single valued on and near a surface $S$, and if $C$ is a closed curve bounding $S$ and positively oriented with respect to $d\mathbf{S}$, then

$$\int_C d\mathbf{s}\times\mathbf{F} = \int_S d\mathbf{S}\times\nabla\times\mathbf{F}$$

If also div **F**, curl **F** both vanish on $S$, prove that $\int_C d\mathbf{S}\times\mathbf{F} = \int_S \frac{\partial\mathbf{F}}{\partial n}\,dS.$ (H.)

11. If **F** is a vector field satisfying curl $\mathbf{F} = 0$, prove that there exists a scalar field $\phi$ such that $\mathbf{F} = \text{grad}\,\phi$.

Referred to orthogonal Cartesian axes $Oxyz$ the components of a vector field **A** are $(yz, zx, xy)$.

Find a vector field **B** such that

$$\text{curl}\,\mathbf{B} = \mathbf{A} \text{ and div}\,\mathbf{B} = 0 \qquad \text{(M.)}$$

# CHAPTER 2

# PHYSICAL FIELDS

The basic formulae for electrostatics, magnetostatics, current flow in continuous media, and gravitational attraction are given in this chapter, and problems in each are worked out. The problems selected are those whose solution involves no special methods—solutions using the complex variable, spherical harmonics, images, and other techniques are given in Volume II.

The fields in all these topics are similar, based on the law of the inverse square, and so the formulae are also very similar. Gravitation problems differ, in that like masses *attract*, whereas in electricity like charges *repel.* The chapter has been split into two sections, each preceded by its own essential information.

2.1 General potential theory, problem 11
Electrostatics, problems 12–21
Magnetostatics, problems 22–29
Current flow, problems 30–31

2.2 Gravitational attraction, problems 32–37

## Section 2.1

**2.1.1 Potential Theory.** (*a*) Uniqueness Theorems.

(i) A three-dimensional vector field which is irrotational, free from discontinuity, and which at infinity is $0(R^{-2})$, must vanish everywhere.

(ii) Hence there is, in a *three-dimensional* region, *only one solution* $\phi$ (apart from a possible arbitrary constant) satisfying $\mathbf{q} = K \operatorname{grad} \phi$, $\operatorname{curl} \mathbf{q} = 0$ with $|\mathbf{q}| = 0(R^{-2})$ at infinity, and $\phi$ or $q_n$ given on any boundaries.

(iii) Similarly there is, in a *two-dimensional finite* region, *only one solution* $\phi$ (apart from a possible arbitrary constant) satisfying $\mathbf{q} = K \operatorname{grad} \phi$, $\operatorname{curl} \mathbf{q} = 0$, with $\phi$ or $q_n$ given on any boundaries.

(iv) But, for a two-dimensional *infinite* region, if $|\mathbf{q}| = 0(R^{-1})$ at infinity, the uniqueness no longer holds.

*N.B.* The forms $|\mathbf{q}| = 0(R^{-2})$, $|\mathbf{q}| = 0(R^{-1})$ at infinity follow directly from Gauss's theorem—see 2.1.2 (12).

### 2.1.2 Table of Basic Formulae

| Electrostatics | Magnetostatics | Current flow |
|---|---|---|
| (1) *Intensity* $\mathbf{E}$ is force on a unit positive charge | *Intensity* $\mathbf{H}$ is force on a unit positive pole | |
| (2) *Potential* $\phi$ at $P$ is work done on a unit positive charge or pole by the field forces in going from $P$ to a fixed point $Q$, at which the potential is taken to be zero | | |
| (3) *A line of force* is a curve whose tangent at any point is in the direction of the intensity at that point. Lines of force cut equipotentials at right angles | | |
| (4) $\mathbf{E} = -\text{grad }\phi$ | $\mathbf{H} = -\text{grad }\phi$ | |
| (5) *Polarisation* $\mathbf{P}$ (see 6) | Magnetisation $\mathbf{I}$ (see 6) | |
| (6) *Displacement* $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ | Magnetic Induction $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{I}$ in magnetised material | Current density $\mathbf{i}$ |
| (7) In isotropic dielectric $\mathbf{D} = K\mathbf{E}$ <br> $K$ is the *dielectric constant* | In material *not permanently magnetised,* and *isotropic* $\mathbf{B} = \mu\mathbf{H}$ <br> $\mu$ is the *magnetic permeability* | $\sigma$ is the conductivity |
| (8) From (4) and (7) $\mathbf{D} = -K \text{ grad }\phi$ | From (4) and (7) $\mathbf{B} = -\mu \text{ grad }\phi$ | If $\phi$ is electrostatic potential, $\mathbf{i} = -\sigma \text{ grad }\phi$ |
| (9) $\text{div }\mathbf{D} = 4\pi\rho$ ($\rho$ volume density of charge) | $\text{div }\mathbf{B} = 4\pi\rho$ ($\rho$ volume density of magnetic poles—imaginary) | |
| (10) $\text{div }\mathbf{D} = 0$ in uncharged space | $\text{div }\mathbf{B} = 0$ in uncharged space, or for any real distribution of *dipoles* | $\text{div }\mathbf{i} = 0$ |
| (11) From (8) and (9) or (10), when $K$ is *constant* $\nabla^2\phi = -4\pi\rho/K$ (Poisson's equation) *or* $\nabla^2\phi = 0$ (Laplace's equation) | From (8) and (9) or (10) when $\mu$ is *constant* $\nabla^2\phi = -4\pi\rho/\mu$ (Poisson's equation) *or* $\nabla^2\phi = 0$ (Laplace's equation) | From (8) and (10) when $\sigma$ is *constant* $\nabla^2\phi = 0$ (Laplace's equation) |
| (12) Gauss's theorem; the flux of intensity across any surface is $4\pi\times$ included charge <br> i.e. $\int_S E_n dS = 4\pi\int \rho dv + 4\pi\Sigma e$ | Gauss's theorem: the flux of intensity across any surface is $4\pi\times$ included poles <br> i.e. $\int_S H_n dS = 4\pi\int \rho dv + 4\pi\Sigma m$ | Gauss's theorem: the flux of current (i.e. total current) across any surface is $4\pi\times$ current strength of included electrodes, i.e. <br> $\int_S i_n\, dS = 4\pi\Sigma J$ * |
| (13) At a *boundary* between two media, (1), (2), where $n_{12}$ is the normal from 1 to 2, (*a*) the tangential component of $\mathbf{E}$, $\mathbf{H}$, is continuous, so $\phi_1 = \phi_2$ | | |
| (*b*) $\mathbf{D}_1 \cdot \mathbf{n}_{12} - \mathbf{D}_2 \cdot \mathbf{n}_{12} = -4\pi\sigma$ where $\sigma$ is surface density of charge <br> $\mathbf{D}_1 \cdot \mathbf{n}_{12} = \mathbf{D}_2 \cdot \mathbf{n}_{12}$ if surface uncharged | $\mathbf{B}_1 \cdot \mathbf{n}_{12} - \mathbf{B}_2 \cdot \mathbf{n}_{12} = -4\pi\sigma$ where $\sigma$ is surface density of poles (imaginary) <br> $\mathbf{B}_1 \cdot \mathbf{n}_{12} = \mathbf{B}_2 \cdot \mathbf{n}_{12}$ for zero surface density or real dipole distribution | $\mathbf{i}_1 \cdot \mathbf{n}_{12} = \mathbf{i}_2 \cdot \mathbf{n}_{12}$ |

* We define $J$, the current strength of an electrode, as $\frac{1}{4\pi}$ times the total current emitted. Other writers use a symbol to denote total current emitted. The object of our notation is to avoid repetition of the factor $4\pi$ and to preserve the analogy with electro- and magneto-statics.

| Electrostatics | Magnetostatics | Current flow |
|---|---|---|
| (14) Some standard potentials and intensities in air, $K = 1$, $\mu = 1$ | | |
| (i) Three-dimensional fields | | |
| (*a*) charge $e$ at the origin | pole $m$ at the origin | Electrode conductivity $\sigma$, current strength $J$, at the origin |
| $\phi = e/r$, $\mathbf{E} = (e/r^3)\mathbf{r}$ | $\phi = m/r$, $\mathbf{H} = (m/r^3)\mathbf{r}$ | $\phi = J/\sigma r$, $\mathbf{i} = (J/r^3)\mathbf{r}$ |
| (*b*) dipole $\mathbf{M}$ at the origin: $\phi = \frac{\mathbf{M} \cdot \mathbf{r}}{r^3}$, $\left.\begin{matrix}\mathbf{E}\\ \mathbf{H}\end{matrix}\right\} = \frac{3(\mathbf{M} \cdot \mathbf{r})}{r^5}\mathbf{r} - \frac{\mathbf{M}}{r^3}$ | | |
| (ii) Two-dimensional fields | | |
| (*a*) Line charge $e$ per unit length at the origin $\phi = -2e \log r$, $\mathbf{E} = (2e/r^2)\mathbf{r}$ | | (*a*) Line electrode strength $J$ per unit length, or circular electrode in flat sheet, at the origin $\phi = \frac{-2J \log r}{\sigma}$, $\mathbf{i} = (2J/r^2)\mathbf{r}$ |
| (15) In uniform dielectric potential change is $\frac{1}{K} \times$ potential change in air | In uniform material, potential change is $\frac{1}{\mu} \times$ potential change in air | |
| (16) $\phi$ is constant on and inside a conductor and there is no included volume charge. Surface charge density is $\sigma = \mathbf{D} \cdot \mathbf{n}/4\pi$. Force on unit surface area is $2\pi\sigma^2\mathbf{n}$ | | $\phi$ is constant on an electrode |
| (17) The *energy* of a set of conductors is $\frac{1}{2}\sum e_i\phi_i$ | | |
| (18) The *energy* of a continuous distribution is $\frac{1}{8\pi}\int_v \mathbf{E} \cdot \mathbf{D}dv$ | The *energy* of a continuous distribution is $\frac{1}{8\pi}\int_v \mathbf{H} \cdot \mathbf{B}dv$ | |

### 2.1.3 Electrostatic Results

(1) The potential due to a uniformly charged sphere, charge $Q$, radius $a$, in air is $\phi = Q/a$, $r<a$; $\phi = Q/r$, $r>a$.

(2) The capacity (i.e. ratio charge : potential) of a spherical condenser, radii $a$, $b$, is $ab/(b - a)$.*

(3) The capacity of a cylindrical condenser, radii $a$, $b$, is $\{2 \log(b/a)\}^{-1}$.

(4) The capacity of a parallel-plate condenser, area $A$, distance $d$, is $A/4\pi d$.

(5) For a set of conductors, the potentials are linear functions of the charges.

* In the special case when the *inner* sphere is earth-connected the capacity is $ab/(b - a) + b$. (See, e.g., Ramsey, *Electricity and Magnetism*, pp. 51–53.)

Thus for a set of $n$ conductors carrying charges $e_1, e_2, \ldots e_n$ the potentials are

$$\begin{aligned}\phi_1 &= p_{11}e_1 + p_{21}e_2 + \ldots p_{n1}e_n\\ \phi_2 &= p_{12}e_1 + p_{22}e_2 + \ldots p_{n2}e_n\\ &\cdots\cdots\cdots\\ \phi_n &= p_{1n}e_1 + p_{2n}e_2 + \ldots p_{nn}e_n\end{aligned}$$

The $p_{ij}$ are called coefficients of potential; $p_{ii}$ depends only on the properties of conductor $i$, and it may be proved that $p_{ij} = p_{ji}$ $(j \neq i)$.

Similarly, the charges $e_i$ may be expressed linearly in terms of the potentials, with coefficients $q_{ij}$, where $q_{ii}$ is called a coefficient of capacity, $q_{ij}$ a coefficient of induction. $q_{ii}$ depends only on the properties of conductor $i$, and $q_{ij} = q_{ji}$.

(6) Green's Reciprocal Theorem: If the charges and potentials on a set of conductors are $e_i$, $\phi_i$ in one state, and $e_i'$, $\phi_i'$ in another state, then

$$\Sigma e_i \phi_i' = \Sigma e_i' \phi_i$$

(7) The lines of force due to a set of collinear point charges $e_i$ are given by

(*a*) in three dimensions: $\Sigma e_i \cos \theta_i =$ constant.
(*b*) in two dimensions: $\Sigma e_i\, \theta_i =$ constant.

where $\theta_i$ is the angle between the join of the point to the $i$th charge and .the line of charges.

### 2.1.4 Magnetic Results

(1) The potential energy of a dipole **M** in a field **H** is $W = -(\mathbf{M} \,.\, \mathbf{H})$.

(2) The force **F** of field **H** on dipole **M** is $\mathbf{F} = (\mathbf{M} \,.\, \text{grad})\, \mathbf{H}$. The couple **G** of field **H** on dipole **M** is $\mathbf{G} = \mathbf{M} \times \mathbf{H}$.

(3) Mutual energy of two dipoles $\mathbf{M}_1$, $\mathbf{M}_2$ at distance $\mathbf{r}_{12}$ is

$$W = (\mathbf{M}_1 \,.\, \mathbf{M}_2)/r_{12}{}^3 - 3(\mathbf{M}_1 \,.\, \mathbf{r}^5{}_{12})\, (\mathbf{M}_2 \,.\, \mathbf{r}_{12})/r^5{}_{12}$$

If the dipoles are coplanar, this reduces to

$$W = M_1M_2/r^3\, (\sin \theta_1 \sin \theta_2 - 2 \cos \theta_1 \cos \theta_2)$$

where $\theta_1$, $\theta_2$ are the angles made by the dipoles with their line join.

(4) The field outside a body, magnetisation **I**, is the same as that due to an (imaginary) volume distribution of density $-$ div **I** in the body and a surface distribution of density **I** . **n**.

### 2.1.5 Current Results

When current flows between two electrodes $A$, $B$, strengths $\pm J$, in a conducting medium, the equivalent resistance, by Ohm's law, is potential difference/total current, i.e. $\dfrac{\phi_A - \phi_B}{4\pi J}$.

**Problem 11**

Prove that the necessary condition for the family of surfaces $f(x, y, z) = \lambda$ (where $\lambda$ is a variable parameter) to be equipotentials in a region of space where there is no charge is that $\dfrac{\nabla^2 f}{(\text{grad}\, f)^2}$ be a function of $\lambda$ only.

In particular, show that the family of cylinders $x^2 + y^2 - 2\lambda x = 0$ can be equipotential surfaces. Determine the corresponding potential and interpret your answer in physical terms. (D.)

---

**Solution.** We must have the potential $\phi = \phi(f)$, since it is to be constant on $f =$ constant.

Then
$$\text{grad}\, \phi = \phi'(f)\,.\,\text{grad}\, f$$
and
$$\nabla^2\phi = \text{div grad}\, \phi = \phi'\nabla^2 f + \phi''\,(\text{grad}\, f)^2$$
In free space
$$\nabla^2\phi = 0$$
And so $\dfrac{\nabla^2 f}{(\text{grad}\, f)^2} = -\dfrac{\phi''}{\phi'}$, a function of $f$, i.e. of $\lambda$ only.

---

For the family
$$x^2 + y^2 - 2\lambda x = 0$$
whence
$$f(x, y, z) = \lambda = \frac{x^2 + y^2}{2x}$$
$\therefore$
$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{x^2 + y^2}{x^3} = \frac{2\lambda}{x^2}$$
and
$$(\text{grad}\, f)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2$$
$$= \left(\frac{1}{2} - \frac{y^2}{2x^2}\right)^2 + \frac{y^2}{x^2}$$
$$= \frac{\lambda^2}{x^2}$$
$\therefore$
$$-\frac{\phi''}{\phi'} = \frac{2}{\lambda}$$
$\therefore$
$$\phi' = \frac{A}{\lambda^2} \text{ and } \phi = B - \frac{A}{\lambda}$$

where $A$ and $B$ are constants.

Thus the potential on the cylinder $x^2 + y^2 - 2\lambda x = 0$ is $\phi = B - A/\lambda$. Eliminating $\lambda$ between these expressions gives for the potential at any point
$$\phi = B - \frac{2Ax}{x^2 + y^2}$$

This is the potential due to a two-dimensional (i.e. line) doublet of moment $A$, at the origin, directed along the $x$-axis. (The constant $B$ is immaterial.)

**Problem 12**

Point charges $e_i$ ($i = 1, 2, \ldots n$) are placed at points $A_i$ along a straight line $XY$. Show that the equations of the lines of force are given by

$$\sum_{i=1}^{n} e_i \cos\theta_i = \text{constant}$$

where $\theta_i$ is the angle made with $XY$ by the radius vector from $A_i$.

Point charges $-e_1$, $e_2$, $-e_1$ are placed at points $A$, $O$, $B$ on a straight line, where $AO = OB$. If $e_2 > 2e_1 > 0$, show that the greatest angle at $O$ which a line of force leaving $O$ and entering $B$ can make with $OB$ is $2\sin^{-1}\sqrt{e_1/e_2}$.

Give a rough sketch of the lines of force. (D.)

---

**Solution.**

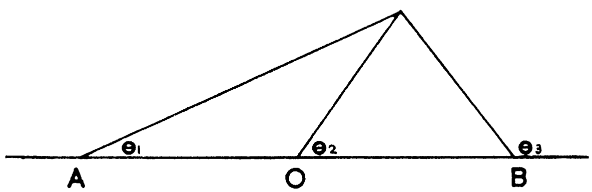


Fig. 5

The equation of a line of force from $O$ to $B$ is

$$-e_1\cos\theta_1 + e_2\cos\theta_2 - e_1\cos\theta_3 = \text{constant (from 2.1.3 (7))}$$

If $\alpha$ is the angle with $OB$ at which the line leaves $O$, $\beta$ that at which it enters $B$ we find for the constant the values:

at $O$, where $\theta_1 = 0,\ \theta_2 = \alpha,\ \theta_3 = \pi,$

$$-e_1 + e_2\cos\alpha + e_1 = \underline{e_2\cos\alpha}$$

at $B$, where $\theta_1 = 0,\ \theta_2 = 0,\ \theta_3 = \beta$

$$-e_1 + e_2 - e_1\cos\beta = \underline{e_2 - e_1 - e_1\cos\beta}$$

Equating these two values of the constant gives

$$e_1(1 + \cos\beta) = e_2(1 - \cos\alpha) \quad . \quad . \quad . \quad . \quad (1)$$

Now $e_2 > 2e_1$, and hence all the lines of force entering at $A$ and $B$ start from $O$. Very close to $B$ the lines of force can enter radially, and so the line of force leaving $O$ so that $\alpha$ is a maximum will have $\beta = 0$. Using this in (1) gives at once the required result.

The limiting line of force leaves $O$ at angle $2\sin^{-1}\sqrt{e_1/e_2}$ and enters $B$ in the direction $BO$.

Lines of force which leave $O$ at a greater angle than this go off to infinity. If the line leaves $O$ at angle $\alpha$, and goes to infinity at angle $\theta$ then

$$(e_2 - 2e_1) \cos \theta = e_2 \cos \alpha$$

i.e.

$$\cos \theta = \frac{e_2}{e_2 - 2e_1} \cos \alpha$$

Thus $\theta < \alpha$, and this holds for $0 \leqslant \cos \alpha \leqslant \dfrac{e_2 - 2e_1}{e_2}$

i.e. for

$$\pi/2 \geqslant \alpha \geqslant 2 \sin^{-1} \sqrt{e_1/e_2}$$

for which the corresponding angles $\theta$ are $\pi/2 \geqslant \theta \geqslant 0$.

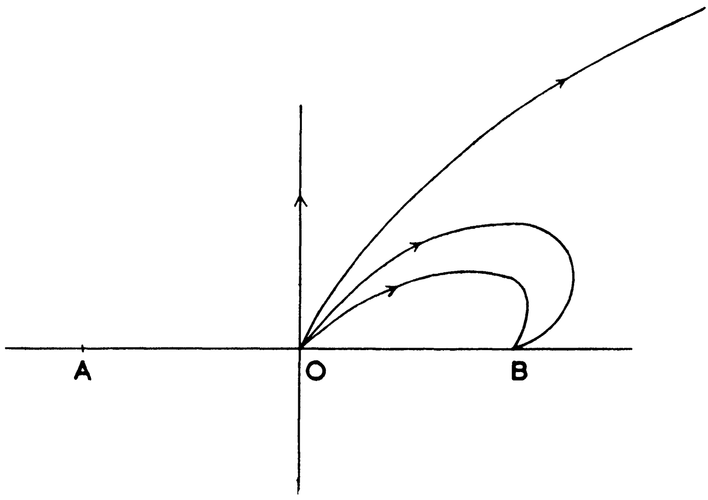


Fig. 6

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**Comment.** The lines of force are almost always best treated in terms of the angles made with *each* of the point charges, rather than in ordinary cartesian or polar coordinates.

---

**Problem 13**

A number of point charges $e_1, e_2, \ldots e_n$ are situated at collinear points $O_1, O_2, \ldots O_n$. If $\theta_r$ is the angle which the line joining $C_r$ to a point $P$ makes with $O_1O_n$, show that for all points $P$ on the same line of force

$$\sum_{r=1}^{n} e_r \cos \theta_r = \text{constant}$$

A point charge $e$ is placed at a point $A$ at a distance $2a$ from the centre $O$ of a spherical conductor of radius $a$ carrying a charge $-e$. Show that the plane perpendicular to $OA$ at a distance $a \cos \alpha$ from $O$ nearer to $A$ will divide the charge on the surface of the conductor into two equal parts if $\alpha$ is given by the equation

$$(2 - \cos \alpha)(5 - 4 \cos \alpha)^{1/2} = 3 \qquad \text{(L.)}$$

---

**Solution.** By the method of images (Vol. II) we can obtain the field outside the charged conductor by replacing it by image charges $-\frac{e}{2}$ at $O$ and $-\frac{e}{2}$ at $A^1$, the inverse point of $A$.

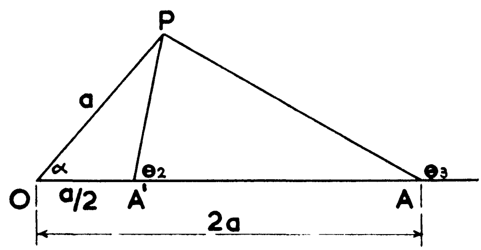


Fig. 7

The lines of force from the charge $e$ at $A$ terminate at $O$ or $A^1$. The problem reduces to finding the angle $AOP$ ($\alpha$), $P$ being the point where the extreme line of force between $A^1$ and $A$ cuts the sphere. By the bookwork, the equation of any line of force between $A^1$ and $A$ is

$$-\tfrac{1}{2} \cos \theta_1 - \tfrac{1}{2} \cos \theta_2 + \cos \theta_3 = \text{constant}$$

If $\beta$ be the angle (with $OA$) at which this line of force enters $A^1$ and $\gamma$ that at which it leaves $A$ we find, for the constant, the values

$$-\tfrac{1}{2} - \tfrac{1}{2} \cos \beta - 1 \text{ and } -\tfrac{1}{2} - \tfrac{1}{2} + \cos \gamma$$

Thus
$$\cos \gamma = -\tfrac{1}{2} \cos \beta - \tfrac{1}{2}$$

But the extreme line of force from $A^1$ to $A$ will have $\beta = 180°$. Hence for this line $\gamma = 90°$ and the equation of this line of force is

$$-\tfrac{1}{2} \cos \theta_1 - \tfrac{1}{2} \cos \theta_2 + \cos \theta_3 = -1$$

If we now put $\theta_1 = \alpha$, so that $P$ is the point on this line of force where it leaves the sphere, we get

$$2 - \cos \alpha = \cos \theta_2 - 2 \cos \theta_3 \quad . \quad . \quad . \quad . \quad (1)$$

By elementary trigonometry and geometry of similar triangles

$$A^1P = \tfrac{1}{2} AP = \frac{a}{1} (5 - 4 \cos \alpha)^{1/2}$$

By projection

$$A^1P\cos\theta_2 - AP\cos\theta_3 = \frac{3a}{2}$$

Substituting and using (1) leads at once to the required result.

---

*Alternatively*, ignoring the hint provided by the bookwork, we may solve the problem by the less subtle, but longer, method of finding the surface density of charge induced on the sphere and then integrating.

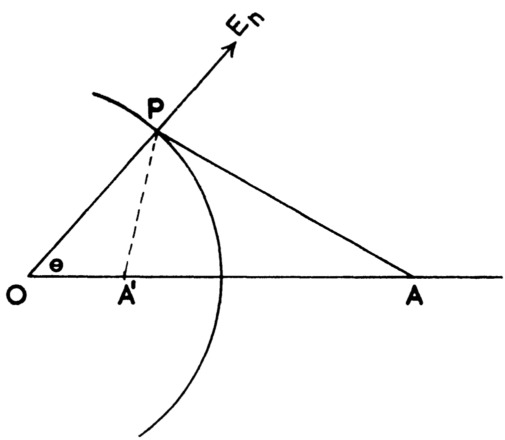


Fig. 8

By easy geometry and trigonometry

$$\left.\begin{aligned}\frac{e}{2}\cos\theta + A^1P\cos A^1\widehat{P}O &= a\\ 2a\cos\theta + AP\cos A\widehat{P}O &= a\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Using, again, the image charges, we have

$$\begin{aligned}E_n &= -\frac{e}{2a^2} - \frac{e}{2A^1P^2}\cos A^1\widehat{P}O + \frac{e}{AP^2}\\ &= -\frac{e}{2a^2} - \frac{ea\,(1-\frac{1}{2}\cos\theta)}{2A^1P^3} + \frac{ea}{AP^3}(1-2\cos\theta)\\ &= -\frac{e}{2a^2} - \frac{4e\,(1-\frac{1}{2}\cos\theta)}{a^2\,(5-4\cos\theta)^{3/2}} + \frac{e\,(1-2\cos\theta)}{a^2(5-4\cos\theta)^{3/2}}\\ &= -\frac{e}{2a^2} - \frac{3e}{a^2\,(5-4\cos\theta)^{3/2}}\end{aligned}$$

and $\sigma = \frac{1}{4\pi}E_n$, from 2.1.2 (16).

If we now integrate over that portion of the surface of the sphere cut off by the cone of semi-vertical angle $\alpha$, vertex $O$, the total charge must be $-\frac{e}{2}$. This leads to

$$\int_0^\alpha \left\{\sin\theta + \frac{6\sin\theta}{(5-4\cos\theta)^{3/2}}\right\} d\theta = 2$$

whence

$$-\cos\alpha + 1 - 3(5 - 4\cos\alpha)^{-1/2} + 3 = 2$$

giving the required result.

---

**Problem 14**

Two wires in the form of circles of radii $a$ and $b$ lie in parallel planes perpendicular to the line joining their centres, which are at a distance $c$ apart. They carry like charges of amounts $e_1$ and $e_2$ respectively. Show that the mutual electrostatic energy of the system is

$$\frac{Ke_1e_2}{\pi\sqrt{ab}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - K^2\sin^2\theta}}.$$

where $K^2 = 4ab\,\{c^2 + (a+b)^2\}^{-1}$.

Find also in the form of an integral the force of repulsion between the wires. (C.)

---

**Solution.**

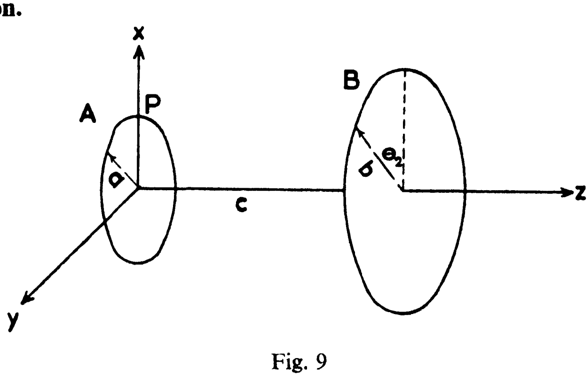


Fig. 9

The potential at a point on either wire due to the charge on the other can be found by integrating the potential due to a point charge, $e/R$, 2.1.2 (14). If axes are taken as shown, the potential at $P(a, 0, 0)$ on wire $A$ due to the element of charge $\frac{e_2}{2\pi}\,d\theta_2$ at $Q(b\cos\theta_2, b\sin\theta_2, c)$ is

$$\frac{e_2}{2\pi}\,\{(b\cos\theta_2 - a)^2 + b^2\sin^2\theta_2 + c^2\}^{-1/2}\,d\theta_2$$

so that, integrating for $\theta_2$ round the wire $B$ from 0 to $\pi$ and doubling, the total potential at $P$ due to charge on $B$ is

$$\frac{e_2}{\pi}\int_0^{\pi}\{a^2+b^2+c^2-2ab\cos\theta_2\}^{-1/2}\,d\theta$$

which, putting $\theta = \pi/2 - \theta_2/2$, and using the expression for $K^2$, reduces to

$$\frac{Ke_2}{\pi\sqrt{ab}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-K^2\sin^2\theta}}$$

By symmetry, this is the potential $\phi_1$ everywhere on wire $A$ due to wire $B$. Similarly, the potential $\phi_2$ everywhere on wire $B$ due to wire $A$ is

$$\frac{Ke_1}{\pi\sqrt{ab}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-K^2\sin^2\theta}}$$

The whole potential $\Phi_1$ of $A$ results from its own charge and the charge on $B$ and is linear in each 2.1.3 (5), i.e.

$$\Phi_1 = p_{11}e_1 + \phi_1$$

and similarly

$$\Phi_2 = p_{22}e_2 + \phi_2$$

where $p_{11}$, $p_{22}$ depend only on $A$ and $B$ respectively.
Hence the *mutual* energy of the system $W$ is $\frac{1}{2}(e_1\phi_1 + e_2\phi_2)$ (2.1.2 (17)),

i.e.
$$W = \frac{Ke_1e_2}{\pi\sqrt{ab}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-K^2\sin^2\theta}}$$

The total energy is $W$ together with the energy of each wire,

i.e.
$$W + \tfrac{1}{2}p_{11}e_1^2 + \tfrac{1}{2}p_{22}e_2^2,$$

and the repulsion between the wires is obtained as usual by differentiating the energy with respect to the distance between them. Since $p_{11}, p_{22}$ are unaffected by this distance, the repulsion is simply $-\partial W/\partial c$,

i.e.
$$\frac{2e_1e_2c}{\pi\{c^2+(a+b)^2\}^{1/2}}\int_0^{\pi/2}\frac{d\theta}{(1-K^2\sin^2\theta)^{3/2}}$$

---

**Problem 15**

In a spherical condenser the inner sphere of radius $a$ is maintained by a battery at constant potential $V$, and the outer concentric thin spherical shell of radius $b$, which is insulated, has total charge $Q$. If the inner sphere expands to radius $c$, show that the energy supplied by the battery is

$$V(V - Q/b)(c - a)$$

Find also the energy supplied by the battery if the inner sphere remains unchanged, radius $a$, and the outer shell expands to radius $d$. (H.)

---

**Solution.** Initially let $e$ be the charge on the inner sphere; the *inside* of the spherical shell then carries charge $-e$, and the *outside* carries charge $Q + e$, since lines of force from the inner sphere must end on the inside of the shell.

The two spheres together form a spherical condenser of capacity $1/a - 1/b$, from 2.1.3 (2), and hence the potential difference is

$$e\left(\frac{1}{a} - \frac{1}{b}\right)$$

But the absolute potential of the shell is known, since on its outer surface it carries a charge $Q + e$ (the charge inside is immaterial, since it is screened). In fact, the shell must be at potential $(Q + e)/b$, from 2.1.3 (1), and so, since the inner sphere is at potential $V$,

$$V - \frac{Q + e}{b} = e\left(\frac{1}{a} - \frac{1}{b}\right) \text{ or } V - \frac{Q}{b} = \frac{e}{a} \quad . \quad . \quad (1)$$

When the inner sphere expands to $c$, there are two effects

(i) the electrical energy increases, and
(ii) mechanical work is done by the force of attraction,

and both these changes have to be supplied by the battery.

(i) To calculate the change in electrical energy we use $\frac{1}{2}\Sigma_1 e\phi$ (2.1.2 (17)), where initially the inner sphere has charge $e$, potential $V$ and outer shell has charge $Q$, potential $\dfrac{Q + e}{b}$, so that the energy is

$$\tfrac{1}{2}\{eV + Q(Q + e)/b\}.$$

Finally, the inner sphere has charge $e^1$, potential $V$ and outer shell has charge $Q$, potential $(Q + e^1)/b$, giving energy $\frac{1}{2}\{e^1 V + Q\,(Q + e^1/b)\}$ where from (1) $\qquad e/a = V - Q/b = e^1/c$
Thus the change in electrical energy is

$$\tfrac{1}{2}\{(e^1 - e)V + (Q/b)(e^1 - e)\} = \tfrac{1}{2}(V + Q/b)(V - Q/b)(c - a)$$

substituting from (1).

(ii) When the radius of the inner sphere is $r$ the charge on it is from (1), $e_r = er/a$, and the outward force is $\int_s 2\pi\sigma^2\, dS = e^2/2a^2$ (2.1.2 (16)), hence the work done is $\int_a^c \frac{1}{2} \cdot \frac{e^2}{a^2}\, dr = \frac{1}{2}(V - Q/b)^2(c - a)$

Thus the total energy produced by the battery is the sum of these, i.e.

$$V(V - Q/b)(c - a).$$

If the inner sphere is unchanged and the outer expands to $d$ we must have, with the notation as before,

$$V - (Q + e')/d = e'\left(\frac{1}{a} - \frac{1}{d}\right), \text{ i.e. } e'/a = V - Q/d \text{ and } e/a = V - Q/b$$

By precisely similar working we find the total energy—mechanical and electrical—in this case to be

$$\frac{aVQ}{bd}(d-b)$$

**Comment.** The two energy requirements, mechanical and electrical, must both be supplied by the battery.

**Problem 16**

Show that if **E** is the electric intensity due to a set of point charges and $S$ is a closed surface in the field, then $\oint \mathbf{E} \cdot d\mathbf{S} = 4\pi Q$, where $Q$ is the sum of the charges enclosed within $S$. Hence obtain Poisson's equation for a volume distribution of charge.

Volume charge is distributed throughout space in such a way that the electrostatic potential at distance $r$ from the origin is given by $e^{-r^3}$. Show that the charge density $\rho$ is given by

$$\rho = \frac{3r(4-3r^3)e^{-r^3}}{4\pi} \qquad \text{(S.)}$$

**Solution.** The potential at any point in space depends only on $r$, i.e. this is a case of spherical symmetry, and Poisson's equation takes the form

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} = -4\pi\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

$$\phi = e^{-r^3}$$

i.e.

$$\frac{d\phi}{dr} = -3r^2e^{-r^3}$$

and

$$\frac{d^2\phi}{dr^2} = -3re^{-r^3}(2-3r^3)$$

and substitution in (1) gives the required value for $\rho$.

**Problem 17**

Show that, in an electrostatic field, the electric intensity just outside a conductor is $4\pi\sigma\mathbf{n}$ where $\sigma$ is the surface density of charge and **n** is a unit vector along the outward normal. Show also that the mechanical force on the surface is $2\pi\sigma^2\mathbf{n}$ per unit area.

A conducting sphere of radius $a$ is electrified to potential $V$. If the sphere consists of two separate hemispheres, show that they repel each other with a force $\frac{1}{8}V^2$; and if the whole is surrounded by an uninsulated concentric spherical conductor of internal radius $b$ and the potential of the solid sphere is still $V$, prove that the force between the hemispheres is increased to

$$\tfrac{1}{8}V^2b^2/(b-a)^2 \qquad \text{(E.)}$$

**Solution.** When the inner sphere is alone it must have a charge $aV$ (from 2.1.3 (1)).

Hence
$$\sigma = \frac{aV}{4\pi a^2} = \frac{V}{4\pi a}$$

Taking, as element of area, a zone bounded by cones of angle $\theta$ and $\theta + \delta\theta$, so that the area of the element is $2\pi a^2 \sin\theta\,\delta\theta$, the component of mechanical force on that element in the direction of the diameter of symmetry is

$$2\pi a^2 \sin\theta\,\delta\theta\,.\,2\pi(V^2/16\pi^2 a^2)\cos\theta$$

The component in the perpendicular direction is clearly zero.

To find the resultant mechanical force on one hemisphere we must integrate between limits 0 and $\pi/2$.

$$\text{Force of repulsion} = \int_0^{\pi/2} \frac{4\pi^2 a^2 V^2}{16\pi^2 a^2} \sin\theta\cos\theta\,d\theta$$
$$= V^2/8$$

When the second conductor is introduced its inner surface forms with the solid sphere a condenser of capacity $\dfrac{ab}{b-a}$ (from 2.1.3 (2)).

$$\text{Charge on inner sphere} = \frac{abV}{b-a}$$

i.e.
$$\sigma = \frac{bV}{4\pi a(b-a)}$$

Proceeding as before, we get for the force between the two hemispheres

$$\frac{1}{4}\frac{b^2V^2}{(b-a)^2}\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta$$
$$= \frac{1}{8}\frac{V^2b^2}{(b-a)^2}$$

---

**Problem 18**

Of two concentric, insulated, thin, conducting, spherical shells, the inner has radius $2a$ and charge $P$, and the outer has radius $4a$ and charge $Q$. The outer shell is coated inside and outside with a layer of thickness $a$ of insulating material with dielectric constant $K$, the remaining spaces being filled with air. Find the potentials of the two shells and show that if the inner one is earthed the potential of the outer becomes

$$\frac{(1 + 6K + 8K^2)Q}{8K(4 + 11K)a} \qquad \text{(D.)}$$

---

**Solution.** Tubes of force from the inner sphere will cause the outer to have a charge $-P$ on its inner surface and $Q - P$ on its outer surface.

If $\phi_1$, $\phi_2$ are the potentials of the inner and outer spheres, respectively,

$$\phi_1 - \phi_2 = \int_{2a}^{3a} P/r^2\, dr + \frac{1}{K}\int_{3a}^{4a} P/r^2\, dr \quad . \quad . \quad . \quad (1)$$

(since the potential drop in dielectric is $1/K$ of the drop in air, from 2.1.2 (15))

and $$\phi_2 = \frac{1}{K}\int_{4a}^{5a} \frac{Q-P}{r^2}\, dr + \int_{5a}^{\infty} \frac{Q-P}{r^2}\, dr \quad . \quad . \quad (2)$$

(2) gives $$\phi_2 = \frac{Q-P}{20aK}(1 + 4K)$$

and then, from (1), we obtain

$$\phi_1 = \frac{Q}{20aK}(1 + 4K) + \frac{P}{30aK}(1 - K)$$

---

After earthing, the inner sphere will have zero potential. Let the charge on the outer sphere be now distributed so that $+e$ is on its inner and $Q - e$ on its outer surface. Let its potential now be $\phi_3$.
There will be an induced charge $-e$ on the inner sphere.

Then $$0 - \phi_3 = \int_{2a}^{3a} -\frac{e}{r^2}\, dr - \frac{1}{K}\int_{3a}^{4a} \frac{e}{r^2}\, dr$$

$$= -\frac{e}{6a} - \frac{e}{12Ka}$$

and $$\phi_3 = \frac{Q-e}{20aK}(1 + 4K)$$

Equating gives $$e = \frac{3(1 + 4K)}{2(4 + 11K)} Q$$

and hence

$$\phi_3 = \frac{(1 + 6K + 8K^2)}{8K(4 + 11K)a}$$

---

**Problem 19**

Two infinite slabs of dielectrics each of unit thickness, but of inductive capacities $K_1$ and $K_2$, have a face in common and separate two infinite, thin plates of conducting material which are maintained at potentials $V_1$ and $V_2$, respectively.

Prove that the potential at any point in the dielectrics is the same as if they were both replaced by an insulated conducting plate charged with surface density $\dfrac{(V_1 - V_2)(K_1 - K_2)}{4\pi(K_1 + K_2)}$ placed in the position of the common face. (L.)

---

**Solution.**

Let surface densities be $\sigma$ and $-\sigma$.
If $\phi_1$ is the potential function in the first slab of dielectric, then

$$-\frac{\partial\phi_1}{\partial x} = \frac{1}{K_1}4\pi\sigma \qquad 0 \leqslant x \leqslant 1$$

(since potential drop in dielectric is $1/K$ of drop in air, from 2.1.2 (15))

Similarly, $$-\frac{\partial\phi_2}{\partial x} = \frac{1}{K_2}4\pi\sigma \qquad 1 \leqslant x \leqslant 2$$

Hence $$\phi_1 = A - \frac{4\pi\sigma}{K_1}x \qquad 0 \leqslant x \leqslant 1$$

$$\phi_2 = B - \frac{4\pi\sigma}{K_2}x \qquad 1 \leqslant x \leqslant 2$$

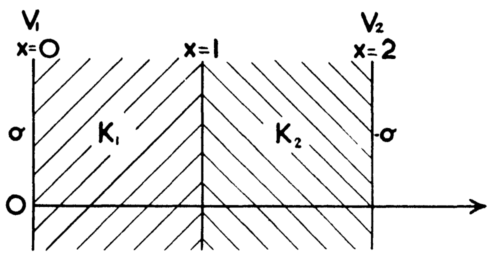


Fig. 10

When $x = 0$, $\phi_1 = V_1$, whence $A = V_1$.
When $x = 2$, $\phi_2 = V_2$, whence $B = V_2 + \frac{8\pi\sigma}{K_2}$
When $x = 1$, $\phi_1 = \phi_2$ whence

$$\sigma = \frac{K_1K_2(V_1 - V_2)}{4\pi(K_1 + K_2)}$$

We thus find for the potentials within the dielectrics

$$\phi_1 = V_1 - \frac{(V_1 - V_2)K_2}{K_1 + K_2}x \quad . \quad . \quad . \quad . \quad . \quad (1)$$

---

$$\phi_2 = V_2 + \frac{2K_1(V_1 - V_2)}{K_1 + K_2} - \frac{K_1(V_1 - V_2)}{K_1 + K_2}x \quad . \quad . \quad (2)$$

---

The dielectric slabs are now removed. Suppose the situation is as shown in the diagram on page 40.
$\sigma_2$ is the net surface density on the middle plate and is thus taken to be

$$\frac{(V_1 - V_2)(K_1 - K_2)}{4\pi(K_1 + K_2)} \qquad \text{as given}$$

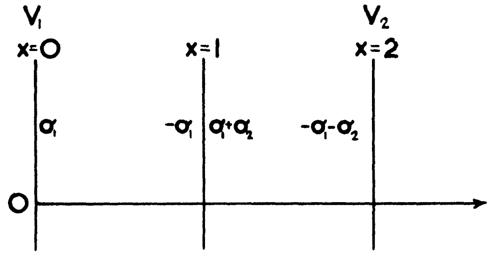


Fig. 11

Proceeding as before we find

$$\phi_1 = L - 4\pi\sigma_1 x \qquad 0 \leqslant x \leqslant 1$$
$$\phi_2 = M - 4\pi(\sigma_1 + \sigma_2)x \qquad 1 \leqslant x \leqslant 2$$

which, after using the boundary conditions $x = 0$, $\phi_1 = V_1$; $x = 1$, $\phi_1 = \phi_2$; $x = 2$, $\phi_2 = V_2$,
and the known value of $\sigma_2$

gives $$L = V_1,\ M = V_2 + 8\pi(\sigma_1 + \sigma_2)$$

and finally $$\sigma_1 = \frac{K_2(V_1 - V_2)}{(K_1 + K_2)}$$

Hence $$\phi_1 = V - \frac{K_2(V_1 - V_2)}{K_1 + K_2}x$$

which is (1) above, and

$$\phi_2 = V + \frac{2K_1(V_1 - V_2)}{K_1 + K_2} - \frac{K_1(V_1 - V_2)}{K_1 + K_2}x$$

which is (2), i.e. the potentials everywhere are the same in both sets of conditions.

---

**Problem 20**

$S_1$ and $S_2$ are two equal fixed insulated conducting spheres. Initially $S_2$ is uncharged, $S_1$ has potential $V$ and charge $P$, and the spheres attract each other with a force $F$. $S_2$ is then raised to a potential $V$ by placing a charge $Q$ on it, and the spheres are found to repel each other with a force $F$. If $S_1$ is now earthed, find the charge induced on it and show that the spheres attract each other with a force

$$\frac{Q(2P^2 - Q^2)}{P^3}F \qquad \text{(D.)}$$

---

**Solution.** Let $p_{11}$, $p_{12}$, $p_{21}$ $(= p_{12})$, and $p_{22}$ $(= p_{11}$ by symmetry, as $S_1$, $S_2$ are spheres of equal radius), be the coefficients of potential of the system, i.e. such that potential $\phi_1 = p_{11}e_1 + p_{12}e_2$, potential

$\phi_2 = p_{21}e_1 + p_{22}e_2$, since the potentials and charges are linearly related (2.1.3 (5)).

| *State 1* | $S_1$ | $S_2$ |
|---|---|---|
| Charge | $P$ | $0$ |
| Potential | $p_{11}P = V$ | $p_{21}P$ |
| Energy | $W_1 = \frac{1}{2}\Sigma_1 e\phi = \frac{1}{2}p_{11}P^2$ | |

Differentiating this to obtain the force between the sphere gives

$$-F = -\frac{\partial W_1}{\partial x} = -\tfrac{1}{2}P^2 p'_{11} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(dashes denoting differentiation with respect to $x$, the distance along line of centres of $S_1$ and $S_2$)

| *State 2* | $S_1$ | $S_2$ |
|---|---|---|
| Charge | $P$ | $Q$ |
| Potential | $p_{11}P + p_{12}Q$ | $p_{11}Q + p_{12}P = V$ |
| Energy | $W_2 = \frac{1}{2}\{p_{11}P^2 + 2p_{12}PQ + p_{11}Q^2\}$ | |

and so $\quad F = -\tfrac{1}{2}\{(P^2 + Q^2)p'_{11} + 2PQp'_{12}\} \quad . \quad . \quad . \quad . \quad (2)$

---

| *State 3* | $S_1$ | $S_2$ |
|---|---|---|
| Charge | $P_1$ | $Q$ |
| Potential | $p_{11}P_1 + p_{12}Q = 0.$ | $p_{11}Q + p_{12}P_1$ |

From State 1, $\quad p_{11} = \dfrac{V}{P}$

From State 2, $\quad p_{12}P = V - p_{11}Q = V - \dfrac{QV}{P}$

so that $\quad p_{12} = \dfrac{V(P - Q)}{P^2}$

From State 3, the charge, $P_1$, induced on $S_1$ is

$$-\frac{p_{12}Q}{p_{11}} = -\frac{VQ(P - Q)}{P^2}\frac{P}{V}$$

$$= -\frac{Q(P - Q)}{P}$$

---

From (1) $\quad p'_{11} = 2F/P^2$

Substituting in (2) gives

$$p'_{12} = -\frac{F}{P^3Q}(2P^2 + Q^2)$$

The energy, $W_3$, in State 3 is

$$\tfrac{1}{2}(p_{11}P_1^2 + 2p_{12}P_1Q + p_{11}Q^2)$$

$$\text{Force between spheres} = -\frac{\partial W_3}{\partial x}$$

$$= -\tfrac{1}{2}\{p'_{11}(P_1{}^2 + Q^2) + p'_{12}P^2Q\}$$

which after reduction gives $-Q\dfrac{(2P^2 - Q^2)}{P^3}F$

---

**Comment.** The solution depends on the linearity of potentials and charges on any system of conductors.

**Problem 21**

Charges $e_1, e_2, \ldots e_n$ in equilibrium on a system of $n$ conductors raise them to potentials $\phi_1, \phi_2, \ldots \phi_n$, while charges $e'_1, e'_2 \ldots e'_n$ would raise the conductors to potentials $\phi'_1, \phi'_2, \ldots \phi'_n$.
Prove that

$$\sum_{r=1}^{n} e_r\phi_r' = \sum_{r=1}^{n} e_r'\phi_r$$

Three conductors, $A_1, A_2, A_3$, are in fixed positions and are originally insulated and uncharged. If any one of the conductors $A_i$ is charged and thereby raised to unit potential the potentials of the other conductors $A_j$ and $A_k$ are found to be $\phi_{ji}$ and $\phi_{ki}$. Prove that

$$\phi_{23}\phi_{31}\phi_{12} = \phi_{32}\phi_{13}\phi_{21}$$

The conductor $A_1$ is given unit charge, the other conductors remaining insulated and uncharged. Each of the conductors $A_2$, $A_3$ and $A_1$ is then earthed and insulated again in turn in this order. Show that the residual charge on $A_1$ is

$$\phi_{12}\phi_{21} + \phi_{13}\phi_{31} - \phi_{23}\phi_{31}\phi_{12} \qquad \text{(L.)}$$

---

**Solution.** The first part of the question is the proof of Green's Reciprocal Theorem.

In the second part, the first state of the field with $A_1$ charged and at unit potential may be shown as

| *State 1* | $A_1$ | $A_2$ | $A_3$ |
|---|---|---|---|
| Charge | $e_1$ | 0 | 0 |
| Potential | 1 | $\phi_{21}$ | $\phi_{31}$ |

and, similarly, we have for the other conductors each charged

| *State 2* | $A_1$ | $A_2$ | $A_3$ |
|---|---|---|---|
| Charge | 0 | $e_2$ | 0 |
| Potential | $\phi_{12}$ | 1 | $\phi_{32}$ |

| *State 3* | $A_1$ | $A_2$ | $A_3$ |
|---|---|---|---|
| Charge | 0 | 0 | $e_3$ |
| Potential | $\phi_{13}$ | $\phi_{23}$ | 1 |

Applying the theorem of the first part (Green's Reciprocal Theorem) to States 1 and 2, then 2 and 3, finally 3 and 1, gives

$$\left.\begin{aligned} e_1\,\phi_{12} &= e_2\,\phi_{21} \\ e_2\,\phi_{23} &= e_3\,\phi_{32} \\ e_3\,\phi_{31} &= e_1\,\phi_{13} \end{aligned}\right\}$$

whence the required result follows.

---

When $A_1$ is given unit charge the state is

| *State 1* | $A_1$ | $A_2$ | $A_3$ |
|---|---|---|---|
| Charge | 1 | 0 | 0 |
| Potential | $\dfrac{1}{e_1}$ | $\dfrac{\phi_{21}}{e_1}$ | $\dfrac{\phi_{31}}{e_1}$ |

When we earth $A_2$ its potential must become zero. It must therefore have acquired a charge sufficient to cancel its potential $\phi_{21}/e_1$, i.e. of amount

$$-e_2\,\frac{\phi_{21}}{e_1}$$

This will alter the potentials of $A_1$ and $A_3$, and so we may show the second state of the field, after $A_2$ has again been insulated, as

| *State 2* | $A_1$ | $A_2$ | $A_3$ |
|---|---|---|---|
| Charge | 1 | $-e_2\,\dfrac{\phi_{21}}{e_1}$ | 0 |
| Potential | $\dfrac{1}{e_1} - \dfrac{\phi_{12}\,\phi_{21}}{e_1}$ | 0 | $\dfrac{\phi_{31}}{e_1} - \dfrac{\phi_{32}\,\phi_{21}}{e_1}$ |

Similarly, after earthing and insulating $A_3$, we have

| *State 3* | $A_1$ | $A_2$ | $A_3$ |
|---|---|---|---|
| Charge | 1 | $-e_2\,\dfrac{\phi_{21}}{e_1}$ | $q_3$ |
| Potential | $\dfrac{1}{e_1} - \dfrac{\phi_{12}\,\phi_{21}}{e_1} + \dfrac{q_3\,\phi_{13}}{e_3}$ | $\dfrac{q_3\,\phi_{23}}{e_3}$ | 0 |

where $q_3$, the charge now on $A_3$, is given by

$$q_3 = -\frac{e_3\,\phi_{31}}{e_1} + \frac{e_3\,\phi_{32}\,\phi_{21}}{e_1}$$

If, after $A_1$ is now earthed and insulated, its residual charge is $q_1$, the final potential of $A_1$ due to all the charges is

$$\frac{q_1}{e_1} - \frac{\phi_{12}\,\phi_{21}}{e_1} + \frac{q_3\,\phi_{13}}{e_3}$$

But this must be zero. Substituting for $q_3$ and using $\phi_{23}\,\phi_{31}\,\phi_{12} = \phi_{32}\,\phi_{13}\,\phi_{21}$, as already proved, gives the required value for $q_1$.

---

**Problem 22**

Find the potential of a field produced by a magnetic dipole of moment **M** and deduce an expression for the magnetic force **H**.

A magnetic field of intensity $H$ and direction $BA$ is applied to two small magnets of moments $M(>0)$ situated at $A$ and $B$ and resting in equilibrium with their axes along $AB$. The magnets are deflected in a plane through small angles $\alpha$ and $\beta$ on the same side of $AB$. Show that their potential energy is approximately

$$M\{(M - \tfrac{1}{2}a^3H)(\alpha^2 + \beta^2) + \alpha\beta M\}/a^3$$

where $a = AB$.

Hence show that both dipoles can rest in stable equilibrium if $Ha^3 < M$. (E.)

**Solution.**

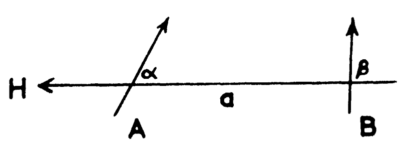


Fig. 12

Since the magnets lie in a plane, at distance $a$ apart, and have moments $M$, their mutual potential energy when deflected through angles $\alpha$, $\beta$ from the line $AB$ is, from 2.1.4 (3),

$$W = -\frac{M^2}{a^3}\,(2\cos\alpha\cos\beta - \sin\alpha\sin\beta)$$

The potential energy of each in the field $H$ along $BA$ is given by the expression, (2.1.4 (1)),

$$W = -(\mathbf{M}\,.\,\mathbf{H})$$

which here gives, for $A$, $+MH\cos\alpha$, and, for $B$, $+MH\cos\beta$.

The total potential energy of the magnets is then

$$W = -\frac{M^2}{a^3}(2\cos\alpha\cos\beta - \sin\alpha\sin\beta) + MH\cos\alpha + MH\cos\beta$$

$$= -\frac{M}{a^3}\{2M(1 - \tfrac{1}{2}\alpha^2)(1 - \tfrac{1}{2}\beta^2) - M\alpha\beta - a^3H(1 - \tfrac{1}{2}\alpha^2) - a^3H(1 - \tfrac{1}{2}\beta^2)\}$$

$$= -\frac{M}{a^3}\{-M(\alpha^2 + \beta^2) - M\alpha\beta + \tfrac{1}{2}a^3H(\alpha^2 + \beta^2) + \text{terms independent of } \alpha,\ \beta\}$$

$$= \frac{M}{a^3}\{(M - \tfrac{1}{2}a^3H)(\alpha^2 + \beta^2) + \alpha\beta M\} \text{ as required.}$$

This expression gives the change in P.E. from the position when the magnets lie along $AB$ to the displaced position $\alpha$, $\beta$; hence, if this is positive for *all* $\alpha$, $\beta$, then the equilibrium position along $AB$ is one of minimum P.E., i.e. one of stable equilibrium.

To investigate this condition, we try to put the expression into the form of a sum of squares. Thus

$$(M - \tfrac{1}{2}a^3H)(\alpha^2 + \beta^2) + \alpha\beta M = \tfrac{1}{2}M(\alpha + \beta)^2 + (\tfrac{1}{2}M - \tfrac{1}{2}a^3H)(\alpha^2 + \beta^2)$$

and the r.h.s. is always positive if $\tfrac{1}{2}M > \tfrac{1}{2}a^3H$, i.e. if $M > Ha^3$ as required.

---

**Problem 23**

Show that the magnetic-field intensity due to a magnetic dipole of moment $\mathbf{M}$ at a field point of position vector $\mathbf{r}$ relative to $\mathbf{M}$ may be written in the form

$$\mathbf{H} = -\frac{\mathbf{M}}{r^3} + 3\frac{(\mathbf{M}\,.\,\mathbf{r})}{r^5}\mathbf{r}$$

Hence find the mutual potential energy of two small magnets.

A small magnet of moment $\mathbf{M}_1$ is fixed in space at a distance $r$ from a second small magnet of moment $\mathbf{M}_2$, which is free to move about an axis at its centre. This axis, the magnetic axis of the first magnet, and the line joining the two magnets form a set of mutually perpendicular lines.

Show that the periodic time for small oscillations of this system is $2\pi(Ir^3/M_1M_2)^{1/2}$, where $I$ is the moment of inertia of the second magnet. (E.)

---

**Solution.**

Since the axis of $M_1$, the line joining $M_1$ and $M_2$, and the direction of rotation of $M_2$ are mutually perpendicular, $M_1$ and $M_2$ are coplanar, in the $x$–$y$ plane. The potential energy then takes the simple form

$$W = -\frac{M_1M_2}{r^3}(2\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \text{ (from 2.1.4. (3))}$$

where $r$ is fixed and $\theta_1 = \pi/2$, which means $W = -\frac{M_1M_2}{r^3}\sin\theta_2$. Thus $\theta_2 = \pm\pi/2$ are the positions of maximum and minimum energy, i.e. of unstable and stable equilibrium respectively.

The equilibrium positions are then when the magnets are parallel; in stable equilibrium the magnets are in *opposite* senses, in unstable equilibrium they are in the *same* sense.

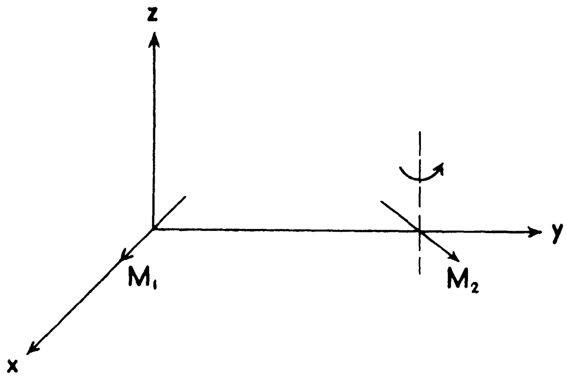


Fig. 13

Consider then oscillations about stable equilibrium, $\theta_2 = -\frac{\pi}{2} + \theta$,

i.e.
$$W = -\frac{M_1M_2}{r^3}\cos\theta$$

Then $\frac{\partial W}{\partial\theta}$ = restoring couple acting $= \frac{M_1M_2}{r^3}\sin\theta = \frac{M_1M_2\theta}{r^3}$ if $\theta$ is small. Hence the magnet swings with an equation of motion

$$I\frac{d^2\theta}{dt^2} = -\frac{M_1M_2}{r^3}\theta,$$

i.e. in S.H.M. with period $2\pi(Ir^3/M_1M_2)^{1/2}$, $I$ being its moment of inertia.

---

**Comment.** Care must be taken to measure both angles in the same sense, between the positive directions of the dipoles and the positive direction of the line joining them.

---

**Problem 24**

Find an expression for the mutual potential energy of two magnetic doublets in the same plane.

Two doublets, each of moment $M_1$, are mounted with their centres at fixed points at distance $R$ apart. They can rotate in the same plane and

are constrained by a frictionless mechanism to be always parallel and in the same sense. Find the positions of equilibrium and discuss their stability.

Prove also that the work which must be done to turn the system from a position of stable to one of unstable equilibrium is $3M^2/R^3$. (H.)

---

**Solution.** The first part is bookwork, and proves that the mutual potential energy of two coplanar magnets, moments $M$, $M^1$, whose axes make $\theta$, $\theta^1$ with their join, is

$$W = -\frac{MM^1}{r^3}(2\cos\theta\cos\theta^1 - \sin\theta\sin\theta^1)$$

For the second part, the magnets are constrained to be always parallel and in the same sense, i.e. so that both make $\theta$ with the join.

The potential energy is

$$W = -\frac{M^2}{R^3}(2\cos^2\theta - \sin^2\theta)$$

The positions of equilibrium are given by $\frac{\partial W}{\partial\theta} = 0$

i.e. by $\quad 4\cos\theta\sin\theta - 2\sin\theta\cos\theta = 0$

i.e. $\quad \theta = 0$ and $\theta = \pi/2$

The second derivative $\frac{\partial^2 W}{\partial\theta^2} = -12\cos 2\theta$, negative for $\theta = 0$, positive for $\theta = \pi/2$

and so $\theta = 0$ is a position of maximum energy, and so *unstable*, and $\theta = \pi/2$ is a position of minimum energy, and so *stable*.

When $\quad \theta = 0,\ W_1 = -\frac{2M^2}{R3}$

When $\quad \theta = \pi/2,\ W_2 = +M^2/R^3$

and hence the work which must be done to turn from a stable to an unstable position is $-(W_1 - W_2)$, i.e. is $3M^2/R^3$.

---

**Problem 25**

Obtain the expression $(\mathbf{M}\,.\,\nabla)\mathbf{H}$ for the mechanical force acting on a magnet of moment $\mathbf{M}$ in a magnetic field $\mathbf{H}$.

Three small magnets of moment $m$, $2m$, and $3m$ are placed respectively at points $(a, 0, 0$, $(0, a, 0)$ and $(0, 0, a)$, and all point towards the origin. Calculate the field at a point $(x, y, z)$ in the neighbourhood of the origin correct to the first order in the coordinates.

A small magnet of moment $M$ is pivoted at the origin of coordinates and points in the direction of the magnetic field. Prove that the resultant force acting on the pivot is of magnitude $\frac{9}{7}\sqrt{35}mM/a^4$. (L.)

---

**Solution.** The field at **R** due to magnet **M** at origin is

$$-\frac{\mathbf{M}}{R^3}+\frac{3(\mathbf{M}\,.\,\mathbf{R})}{R^5}\,\mathbf{R}\ \text{(from 2.1.4 (3))}$$

For the field at $\mathbf{r} = (x, y, z)$ due to the three given magnets at positions **A, B, C** we may substitute in this expression

$$\begin{aligned}\mathbf{M}_1 &= (-m, 0, 0,),\ \mathbf{R}_1 = \mathbf{r} - \mathbf{A} = (x-a, y, z)\\ \mathbf{M}_2 &= (0, -2m, 0),\ \mathbf{R}_2 = \mathbf{r} - \mathbf{B} = (x, y-a, z)\\ \mathbf{M}_3 &= (0, 0, -3m),\ \mathbf{R}_3 = \mathbf{r} - \mathbf{C} = (x, y, z-a)\end{aligned}$$

giving a field of

$$-\frac{\mathbf{M}_1}{\{(x-a)^2+y^2+z^2\}^{3/2}}+\frac{3\{-m(x-a)\}\mathbf{R}_1}{\{(x-a)^2+y^2+z^2\}^{5/2}}$$

with two other similar expressions.
To the first order in $x$, $y$, $z$ this is

$$-\frac{\mathbf{M}_1}{a^3}\left(1+\frac{2x}{a}\right)+\frac{3}{a^5}\left(1+\frac{5x}{a}\right)\{-m(x-a)\}\mathbf{R}_1$$

which gives finally the three components

$$-\frac{2m}{a^3}\left(1+\frac{3x}{a}\right),\ \frac{3my}{a^4},\ \frac{3mz}{a^4}$$

Similarly, we have $\dfrac{6mx}{a^4},\ -\dfrac{4m}{a^3}\left(1+\dfrac{3y}{a}\right),\ \dfrac{6mz}{a^4}$

and $\dfrac{9mx}{a^4},\ \dfrac{9my}{a^4},\ -\dfrac{6m}{a^3}\left(1+\dfrac{3z}{a}\right)$

which sum to $\dfrac{m}{a^3}\left(-2+\dfrac{9x}{a}\right),\ -\dfrac{4m}{a^3},\ \dfrac{m}{a^3}\left(-6-\dfrac{9z}{a}\right)$

These are the components of **H**, the magnetic field.

At the origin, $\mathbf{H}_0 = -2m/a^3, -4m/a^3, -6m/a^3$, i.e. has direction cosines $(-1/\sqrt{14}, -2/\sqrt{14}, -3/\sqrt{14})$. A magnet of moment $M$ lies in this direction, i.e. has components $-M/\sqrt{14}, -2M/\sqrt{14}, -3M/\sqrt{14}$. By the first part, the mechanical force on this magnet—i.e. on the pivot, since the magnet is pivoted—is $(\mathbf{M}\,.\,\nabla)\mathbf{H}$

i.e. $$-\frac{Mm}{\sqrt{14}a^3}\left(\frac{\partial}{\partial x}+2\frac{\partial}{\partial y}+3\frac{\partial}{\partial z}\right)\left\{\left(-2+\frac{9x}{a}\right), -4, \left(-6-\frac{9z}{a}\right)\right\}$$

i.e. $$-\frac{Mm}{\sqrt{14}a^3}(9/a, 0, -27/a)$$

and so has magnitude

$$9\frac{Mm}{\sqrt{14}a^4}\sqrt{10}=\frac{9}{7}\frac{Mm}{a^4}\sqrt{35}$$

**Comment.** A direct application of the theory, with a simplification introduced by evaluating the field approximately in the neighbourhood of the origin.

---

**Problem 26**

Show that the magnetic field at a point $P$ due to a small magnet of moment $\mathbf{m}$ with its centre at $O$ is

$$\frac{3(\mathbf{m}\,.\,\mathbf{r})}{r^5}\,\mathbf{r} - \mathbf{m}/r^3$$

where $\mathbf{r} = \overline{OP}$ and $r = |\mathbf{r}|$.

A small magnet of moment $m\mathbf{j}$ is fixed at the point $a\mathbf{i}$ and another of moment $m\mathbf{i}$ at the point $a\mathbf{j}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors along the axes of a rectangular set $Oxyz$. A third small magnet is pivoted at the point $a\mathbf{k}$ so that it can turn freely about its centre. Show that there are two positions of equilibrium of this magnet both lying in the plane $z = a$ and distinguish between the stable and unstable positions.

If a magnetic field $H\mathbf{k}$ is superimposed on the system, where $H$ is constant, find the direction of the movable magnet in the position of stable equilibrium. (L.)

---

**Solution.** Magnet $M_1$ at $a\mathbf{i}$, moment $m\mathbf{j}$, has field at $a\mathbf{k}$ of

$$\frac{3\{m\mathbf{j}\,.\,(a\mathbf{k} - a\mathbf{i})\}}{r^5}(a\mathbf{k} - a\mathbf{i}) - \frac{m\mathbf{j}}{r^3} \text{ which reduces to } \frac{-m\mathbf{j}}{2\sqrt{2}a^3}$$

since $$\mathbf{j}\,.\,\mathbf{k} = \mathbf{j}\,.\,\mathbf{i} = 0$$

Similarly, $M_2$ at $a\mathbf{j}$, moment $m\mathbf{i}$, has field at $a\mathbf{k}$ of

$$-\frac{m\mathbf{i}}{2\sqrt{2}a^3}$$

The total field at $a\mathbf{k}$ due to $M_1$ and $M_2$ is

$$-\frac{m}{2\sqrt{2}a^3}(\mathbf{i} + \mathbf{j})$$

There are two positions of equilibrium for the magnet $M_3$ along the field, i.e. in the plane $z = a$. In the stable position $M_3$ lies along the field, in the unstable position against the field.

If a magnetic field $H\mathbf{k}$ is superimposed the total field at $a\mathbf{k}$ is

$$-\frac{m}{2\sqrt{2}a^3}(\mathbf{i} + \mathbf{j}) + H\mathbf{k}$$

which gives the direction of the movable magnet in stable equilibrium.

---

**Problem 27**

Show that the scalar magnetic potential at any point due to a mass of magnetised matter occupying a volume $V$ can be expressed in terms of a fictitious volume density $\rho$ of magnetic poles distributed throughout $V$ and of a fictitious surface density $\sigma$ of magnetic poles distributed over the surfaces bounding $V$.

The interior of the circular cylinder $x^2 + y^2 = a^2$ is occupied by magnetised material, the intensity of magnetisation being $(py + qa, px, 0)$ where $p$ and $q$ are constants. Show that the magnetic scalar potential at the point $(x, y)$ inside the cylinder is $2\pi x(py + qa)$ and find the corresponding expression for the potential at an external point.

(H.)

---

**Solution.** The first part is bookwork, and shows that the magnetic field $\mathbf{I}$ is equivalent to the field produced by a volume density $-\operatorname{div} \mathbf{I}$ and surface density $\mathbf{I} \cdot \mathbf{n}$ of magnetic poles.

Given that, within $x^2 + y^2 = a^2$,

$$\mathbf{I} = (py + qa, px, 0)$$

and outside $\mathbf{I} = 0$, then, by differentiation, $-\operatorname{div} \mathbf{I} = 0$ everywhere; and, on the surface, $\mathbf{n}$ is $(x/a, y/a, 0)$

so that $\mathbf{I} \cdot \mathbf{n} = 2pxy/a + qx$

Thus the magnetic field is that produced by a surface density

$$\sigma = 2pxy/a + qx.$$

We then require potential functions $\phi_1$ (inside) and $\phi_2$ (outside) the cylinder, satisfying

(i) $\nabla^2\phi_1 = \nabla^2\phi_2 = 0$

$$\left.\begin{aligned} &\text{(ii) } \phi_1 = \phi_2 \text{ on the cylinder} \\ &\text{(iii) } \frac{\partial\phi_1}{\partial n} - \frac{\partial\phi_2}{\partial n} = 4\pi\left(\frac{2pxy}{a} + qx\right) \end{aligned}\right\} \text{from 2.1.2 (13)}$$

(iv) $\phi_1$ continuous at $x = y = 0$

(v) $\phi_2 \longrightarrow 0$ as $x, y \longrightarrow \infty$

The form of (iii) suggests trying $\phi_1 = Axy + Bx$ (since this will give terms in $xy$ and $x$ in $\dfrac{\partial\phi}{\partial n}$ on the boundary) and, for $\phi_2$, we may use the fact (Vol. II—Special Methods) that if a homogeneous function $\phi$ of order $n$ in two variables $x, y$ satisfies $\nabla^2\phi = 0$ so does the function $\phi/r^{2n}$, where $r^2 = x^2 + y^2$.

Thus for the outer potential, we may take $\phi_2 = Aa^4xy/r^4 + Ba^2x/r^2$, in which form (ii) and (v) are satisfied.

$A$ and $B$ are then found from (iii), which gives

$$(2Axy + Bx) - \left(2Axy + Bx - \frac{4A}{a}xy - \frac{2B}{a}x\right) = 4\,(2pxy/a + qx)$$

and, equating terms in $xy$ and $x$,

$$A = 2\pi p,\ B = 2\pi aq$$

Thus $$\phi_1 = 2\pi x(py + aq)$$

and $$\phi_2 = 2\pi x(pa^4y/r^4 + a^3q/r^2)$$

**Comment.** A method, which is covered thoroughly in Vol. II, of building up a potential from known functions in order to satisfy given conditions.

## Problem 28

A sphere has uniform intensity of magnetisation **I**. A cubical cavity (not necessarily small) having faces perpendicular to the direction of **I** is excavated from the sphere. By making use of Poisson's equivalent distribution of magnetic poles, or otherwise, prove that the magnetic force at the centre of the cavity is zero.

Hence, or otherwise, prove that the magnetic force is zero at the centre of *any* cubical cavity excavated from the sphere. (H.)

**Solution.** For a uniformly magnetised body, since div **I** = 0, Poisson's magnetic distribution reduces to a surface distribution of magnetic poles of density **I** . **n**.

For a cubical cavity, edge $a$, with two faces perpendicular to **I** (and the other four parallel to **I**) this gives:

($a$) distributions of density $I$, $-I$ over the 2 end faces (where the normals are parallel to **I** and in opposite directions);

($b$) zero distribution over the side faces (normals perpendicular to **I**).

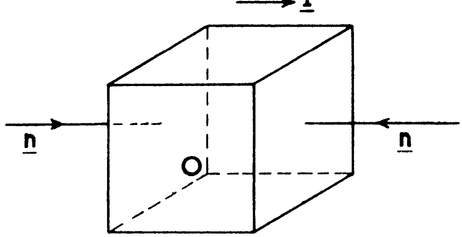


Fig. 14

The centre of the cavity, $O$, is then midway between two parallel sheets of equal and opposite magnetic poles : and so the magnetic force at $O$ is zero.

Now considering *any* cubical cavity excavated from the sphere, let the components of the magnetisation **I** perpendicular to each pair of forces of the cube be $I_1$, $I_2$, $I_3$.

By applying the above result, the component $I_1$ is equivalent to equal and opposite pole distributions on the two corresponding faces, and similarly $I_2$ and $I_3$ on the other pairs of faces.

But for each pair of faces, the field at the centre of the cavity is zero, and hence the total field there is zero.

**Problem 29**

Obtain the conditions satisfied by a magnetic field at a surface separating two media of different permeabilities.

An infinitely long hollow cylinder of permeability $\mu$ whose cross-section referred to polar coordinates $(r, \theta)$ is bounded by $r = a$, $r = b (a > b)$ is placed in a uniform magnetic field of intensity $H$ acting parallel to the direction of the initial line.

Show that in the hollow of the cylinder the magnetic field is uniform and of intensity

$$4H\mu a^2/\{a^2(\mu + 1)^2 - b^2(\mu - 1)^2\}$$

Prove also that the lines of magnetic induction within the material of the cylinder are given by the equation

$$\left\{r - \frac{(\mu - 1)}{(\mu + 1)}\frac{b^2}{r}\right\} \sin\theta = \text{constant} \qquad \text{(C.)}$$

**Solution.**

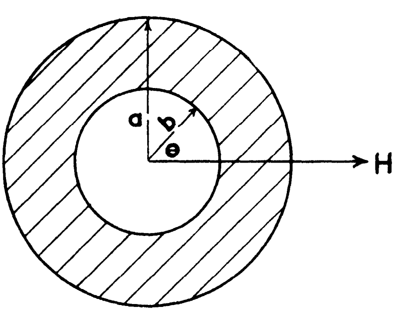


Fig. 15

We require potentials

$\phi_0$ outside the cylinder $a < r$
$\phi_m$ within the cylinder $b < r < a$
$\phi_i$ inside the cylinder $r < b$

satisfying the conditions

(i) $\nabla^2\phi = 0$ everywhere

(ii) $\phi$ continuous on all boundaries

(iii) $\mu\dfrac{\partial\phi}{\partial n}$ continuous on all boundaries

(ii)–(iii) from 2.1.2 (13)

(iv) $\phi_0 \longrightarrow -Hr\cos\theta$ as $r \longrightarrow \infty$

(v) $\phi_i$ finite at the origin

To satisfy (i) and (iv), we try

$$\phi_0 = -Hr\cos\theta + \frac{A\cos\theta}{r} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

since these are known to be solutions of $\nabla^2\phi = 0$ in two dimensions;

then $\quad \left(\dfrac{\partial\phi_0}{\partial n}\right)_{r=a} = \left(\dfrac{\partial\phi_0}{\partial r}\right)_{r=a} = -H\cos\theta - \dfrac{A\cos\theta}{a^2}$

Thus we want $\quad \left(\mu\dfrac{\partial\phi_m}{\partial r}\right)_{r=a} = -H\cos\theta - \dfrac{A\cos\theta}{a^2}$

so try $$\phi_m = Br\cos\theta + \frac{C\cos\theta}{r} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and, because of (v), $\quad \phi_i = Er\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$

and then $\quad \mu B - \mu C/a^2 = -H - A/a^2 \quad . \quad . \quad . \quad . \quad . \quad (4)$

$\quad \mu B - \mu C/b^2 = E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$

and, from (1) and (2), $B + C/a^2 = -H + A/a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$

from (2) and (3) $\quad B + C/b^2 = E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$

Then (4), (5), (6), and (7) can be solved to give

$$\left.\begin{aligned} B\{a^2(\mu+1)^2 - b^2(\mu-1)^2\} &= -2Ha^2(\mu+1) \\ C\{a^2(\mu+1)^2 - b^2(\mu-1)^2\} &= -2Ha^2b^2(\mu-1) \end{aligned}\right\} \quad . \quad (8)$$

Hence in the hollow the field $-\text{grad}\,\phi_i$ is uniform in the direction of $H$ and, from (8) and (5), is of magnitude

$$-E = 4H\mu a^2\{a^2(\mu+1)^2 - b^2(\mu-1)^2\}^{-1}$$

as required.

Within the material of the cylinder, the magnetic induction $\mathbf{B} = -\mu\,\text{grad}\,\phi_m$ and lines of magnetic induction are lines whose tangents are everywhere parallel to $\mathbf{B}$.

From (2) $$\phi_m = Br\cos\theta + \frac{C\cos\theta}{r}$$

Hence, using (8),

$$\phi_m = -2Ha^2(\mu-1)\left\{r\cos\theta + \frac{b^2}{r}\left(\frac{\mu-1}{\mu+1}\right)\cos\theta\right\}$$
$$\{a^2(\mu+1)^2 - b^2(\mu-1)^2\}^{-1}$$

Hence the induction line has equation

$$dr : rd\theta = \frac{\partial \phi_m}{\partial r} : \frac{1}{r}\frac{\partial \phi_m}{\partial \theta}$$

i.e. $$r\cos\theta\left\{1 - \frac{b^2}{r^2}\left(\frac{\mu - 1}{\mu + 1}\right)\right\} d\theta + \sin\theta\left\{1 + \frac{b^2}{r^2}\left(\frac{\mu - 1}{\mu + 1}\right)\right\} dr = 0$$

and this integrates immediately, to give

$$\sin\theta\left\{r - \frac{b^2}{r}\left(\frac{\mu - 1}{\mu + 1}\right)\right\} = \text{constant}$$

which is the required equation for a line of magnetic induction within the cylinder.

---

**Comment.** Another example (like Problem 27) of the technique, covered thoroughly in Vol. II, of building up a potential from known functions to satisfy given conditions.

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**Problem 30**

Show that if $\phi$ is the potential at any point of a medium of conductivity $\sigma$ in which steady currents flow, then div ($\sigma$ grad $\phi$) $= 0$.

A block of conducting material is in the form of a cube of side $a$, and the conductivity at any point in the block is $\lambda(a + x)$, where $x$ is the distance of the point from a face $S$ of the cube and $\lambda$ is a constant. Show that the resistance of the block between $S$ and the opposite face is $\frac{1}{\lambda a^2}\log 2$ and find the resistance between a pair of opposite faces perpendicular to $S$. (It may be assumed that the currents are uni-directional in each case.) (L.)

---

**Solution.**

From the bookwork of the first part

$$\operatorname{div}\{\lambda(a + x)\operatorname{grad}\phi\} = 0$$

or $$\lambda(a + x)\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) + \lambda\frac{\partial\phi}{\partial x} = 0 \quad . \quad (1)$$

When current flows from $S$ to the opposite face there is no potential drop between any other pair of opposite faces (else current would flow between such a pair) and hence

$$\frac{\partial^2\phi}{\partial y^2} = \frac{\partial^2\phi}{\partial z^2} = 0 \text{ (since } \phi \text{ is a function of } x \text{ only)}$$

so from (1), $$\lambda(a + x)\frac{d^2\phi}{dx^2} + \lambda\frac{d\phi}{dx} = 0$$

whence $\phi = A\log(x + a) + B$, where $A$ and $B$ are constants.

Thus the potential difference between $S(x = 0)$ and the opposite face $(x = a)$ is given by

$$\phi_a - \phi_0 = A \log 2$$

Also the total current flowing is, by 2.1.2 (8) and (12),

$$I = a^2 \sigma \left|\frac{d\phi}{dx}\right|$$

$$= a^2 \lambda A$$

and the resistance $= \dfrac{\phi_a - \phi_0}{I} = \dfrac{1}{\lambda a^2} \log 2$

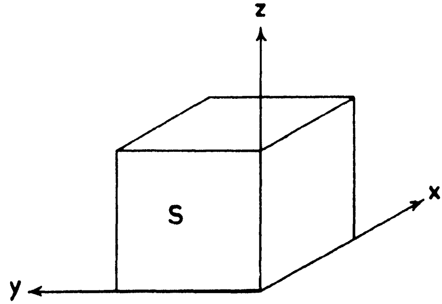


Fig. 16

We now consider current flowing between the faces $y = 0$ and $y = a$. Then $\phi$ is a function of $y$ only and (1) becomes

$$\frac{d^2\phi}{dy^2} = 0$$

$\therefore$ $$\phi = Ay + B$$

so that the potential difference between the two faces is $Aa$. To find the total current flowing we must evaluate

$$\iint \sigma \left|\frac{d\phi}{dy}\right| dx\, dz \text{ [2.1.2 (8) and (12)]}$$

over any face $y = k \quad (0 \leqslant k \leqslant a)$. This is then

$$\int_0^a \int_0^a \lambda(x + a)A\, dx\, dz = \tfrac{3}{2}\lambda A a^3$$

giving resistance $= 2/(3\lambda a^2)$

---

**Problem 31**

Two small, spherical, perfectly conducting electrodes of radii $\delta$ and $\delta^1$ are embedded in an infinite medium of conductivity $\sigma$, their centres

being at a distance $2a$ apart. Show that the resistance between them is approximately

$$R = \frac{1}{4\pi\sigma}\left(\frac{1}{\delta} + \frac{1}{\delta^1} - \frac{1}{a}\right)$$

If the medium is bounded by an infinite plane surface from which each of the centres of the electrodes is at distance $b$, show that the resistance

is $$R + \frac{1}{4\pi\sigma}\left(\frac{1}{b} - \frac{1}{\sqrt{(a^2 + b^2)}}\right) \qquad \text{(C.)}$$

**Solution.**

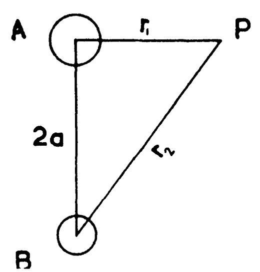


Fig. 17

Let $A$ be the electrode of radius $\delta$ and strength $J$ through which current enters and $B$ the other where the current leaves, i.e. of strength $-J$. At any point $P$, with $AP = r_1$, $BP = r_2$, we have the potential

$$\phi_P = \frac{J}{\sigma r_1} - \frac{J}{\sigma r_2}, \text{ by 2.1.2 (14),}$$

At $A$, $$r_1 = \delta, \; r_2 \simeq 2a$$

$\therefore$ $$\phi_A \simeq \frac{J}{\sigma}\left(\frac{1}{\delta} - \frac{1}{2a}\right)$$

Similarly $$\phi_B \simeq \frac{J}{\sigma}\left(\frac{1}{2a} - \frac{1}{\delta^1}\right)$$

so resistance $= \dfrac{\phi_A - \phi_B}{4\pi J} = \dfrac{1}{4\pi\sigma}\left(\dfrac{1}{\delta} + \dfrac{1}{\delta^1} - \dfrac{1}{a}\right)$ (from 2.1.5.)

To solve the second part we use the method of images.

At the image point, $A^1$, of $A$ in the boundary introduce a similar electrode of strength $J$, and likewise at $B^1$ an electrode similar to that at $B$ with strength $-J$. The choice of currents in these electrodes is made to ensure that the current density $i_n$ is zero across the boundary.

Then at a point $P$ in the medium we have

$$\phi_P = \frac{J}{\sigma}\left(\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_1'} - \frac{1}{r_2'}\right)$$

Hence at $A$ and $B$

$$\phi_A \simeq \frac{J}{\sigma}\left(\frac{1}{\delta} - \frac{1}{2a} + \frac{1}{2b} - \frac{1}{2\sqrt{(a^2+b^2)}}\right)$$

$$\phi_B \simeq \frac{J}{\sigma}\left(\frac{1}{2a} - \frac{1}{\delta^1} + \frac{1}{2\sqrt{(a^2+b^2)}} - \frac{1}{2b}\right)$$

$\therefore$ $$\text{resistance} = \frac{\phi_A - \phi_B}{4\pi J}$$

$$= R + \frac{1}{4\pi\sigma}\left(\frac{1}{b} - \frac{1}{\sqrt{(a^2+b^2)}}\right)$$

---

**Comment.** The method of images is covered fully in Vol. II.

## 2.2 Gravitation Results

(1) The *intensity* $\mathbf{F}$ is the force on unit mass.

(2) The *potential* $\phi$ at $P$ is the work done by the field in moving unit mass $P$ from a fixed point $Q$, at which the potential is taken to be zero. $Q$ is generally taken to be at infinity.*

(3) $$\mathbf{F} = \text{grad } \phi$$

(4) $$\text{div } \mathbf{F} = -4\pi\gamma\rho, \ \gamma \text{ the gravitational constant.}$$

Hence $$\nabla^2\phi = -4\pi\gamma\rho \text{ (Poisson's equation)}$$

(5) At a boundary between two media, 1, 2,

(a) $\phi_1 = \phi_2$

(b) $\mathbf{F}_1 . \mathbf{n}_{12} - \mathbf{F}_2 . \mathbf{n}_{12} = 4\pi\gamma\sigma$

(6) The gravitational energy $W$ is the work done against the attractive forces in collecting the particles from a state of infinite dispersion.

For a discrete distribution $$W = -\tfrac{1}{2}\sum m_i\phi_i$$

For a continuous distribution $$W = -\tfrac{1}{2}\int \rho\phi \, dv$$

(7) *Attraction and Potential of a Uniform Straight Rod*

If $\rho$ is the density per unit length of a uniform rod $AB$, then;

(i) The attraction at a point $P$, not in the line of the rod, is

$$\frac{2\gamma\rho \sin \alpha}{p}$$

where $2\alpha$ is the angle subtended by $AB$ at $P$ and $p$ is the length of the perpendicular from $P$ to $AB$ (or $AB$ produced). A convenient form

* Some authorities define $\phi$ to be minus this value; in this case the signs of terms involving $\phi$ in the results quoted must be altered.

for the component of attraction *parallel* to $AB$ (and in the same sense) is

$$\gamma\rho\left(\frac{1}{PA}-\frac{1}{PB}\right)$$

(ii) The potential at $P$ is $\gamma\rho \log \dfrac{r_1+r_2+a}{r_1+r_2-a}$

where $AB = a$, $AP = r_1$, $BP = r_2$

(8) *Attraction and Potential of a Uniform Plane Circular Lamina at a point P on the Axis of Symmetry*

(i) Attraction $= 2\pi\gamma\rho\,(1 - \cos\alpha)$, where $\alpha$ is the semi-vertical angle of the cone with the lamina as base and $P$ as vertex.

This result can be obtained from first principles by integration, but follows at once from the following general theorem:

*Theorem.* The resolved part of the attraction, at a point $P$, of a thin uniform plane lamina, surface density $\rho$, in the direction perpendicular to the lamina is $\gamma\rho\omega$, where $\omega$ is the solid angle subtended by the lamina at $P$.

(ii) Potential $= 2\pi\gamma\rho\{\sqrt{r^2+p^2} - p\}$
where $r$ is the radius of the disk and $p$ the perpendicular distance of $P$ from it.

(9) *Attraction and Potential of a Uniform, Thin, Spherical Shell*

(i) Attraction

$$\begin{aligned} &= \gamma M/r^2 \qquad r > a \\ &= 0 \qquad\qquad r < a \end{aligned}$$

where $M$ is the mass, $a$ the radius of the shell, and $r$ the distance of the point from the centre of the shell.

This follows at once from *Gauss's Theorem*, which may be stated as:

The *outward* flux of the force of attraction over any closed surface in a gravitational field of force is equal to $-4\pi\gamma$ times the mass enclosed by the surface.

(ii) Potential

$$\begin{aligned} &= \frac{\gamma M}{r} \qquad (r > a) \\ &= \frac{\gamma M}{a} \qquad (r \leqslant a) \end{aligned}$$

(10) *Attraction and Potential of a Uniform Solid Sphere*

(i) By integrating the result of (9)(i), or directly by Gauss's Theorem, the attraction at a point distant $r(\geqslant a)$ from the centre can be shown to be

$$\frac{\gamma M}{r^2}$$

(ii) Potential

$$= \frac{\gamma M}{r} \qquad (r \geqslant a)$$

**Problem 32**

A uniform straight rod is of mass $m$ per unit length. Find the components along and perpendicular to the rod of the gravitational field at any point.

A uniform rectangular plate of surface density $\sigma$ per unit area has sides of lengths $a$ and $b$. A point $P$ is a distance $c$ along the perpendicular at one corner. Show that the gravitational field at $P$ parallel to the sides of length $a$ is

$$\gamma\sigma \log\left[\frac{(a^2+c^2)^{1/2}}{c} \cdot \frac{(b^2+c^2)^{1/2}+b}{\{(a^2+b^2+c^2)^{1/2}+b\}}\right]$$

where $\gamma$ is the gravitational constant. (D.)

**Solution.**

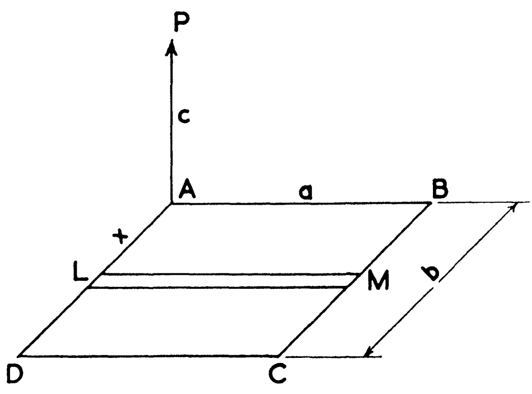


Fig. 18

Consider the component of intensity of attraction at $P$, parallel to $AB$, due to an elemental strip, $LM$, of width $\delta x$, and hence of density $\sigma\delta x$ per unit length, where $AL = x$.

The bookwork of the first part gives this component as

$$\gamma\sigma\delta x\left(\frac{1}{PL} - \frac{1}{PM}\right)$$

This at once leads to the value of the component due to the whole plate as

$$\gamma\sigma\int_0^b \{(c^2+x^2)^{-1/2} - (c^2+a^2+x^2)^{-1/2}\}\,dx$$

$$= \gamma\sigma\left\{[\log(x+\sqrt{c^2+x^2})]_0^b - [\log(x+\sqrt{c^2+a^2+x^2})]_0^b\right\}$$

giving the result stated.

**Problem 33**

Two uniform straight rods each of length $l$ are placed parallel to each other and perpendicular to the line joining their centres, which are at a distance $h$ apart. If their masses are $M$, $N$ respectively, show that the force of attraction between them is

$$\frac{2\gamma MN}{l^2}\left\{\left(l+\frac{l^2}{h^2}\right)^{1/2}-1\right\}$$

where $\gamma$ denotes the gravitational constant. (D.)

**Solution.**

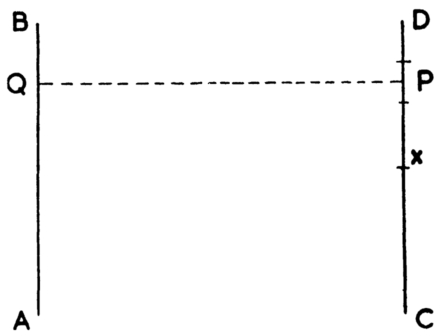


Fig. 19

The density, $\rho_1$, of $AB$ is $\frac{M}{l}$, and of $CD$ is $\rho_2 = \frac{N}{l}$.

Measuring $x$ from the mid-point of rod $CD$, consider an element of length $\delta x$ at $P$. The force of attraction between this element and rod $AB$ is directed along the bisector of $BPA$ and is of magnitude

$$\frac{2\gamma\rho_1\rho_2 \sin\alpha\,\delta x}{h} \text{ (from 2.2 (7))}$$

Thus the total force of attraction between the rods is

$$2\int_0^{l/2}\frac{2\gamma\rho_1\rho_2}{h}\sin\alpha\cos(\alpha-\theta)\,dx$$

where $\theta = B\hat{P}Q$, $PQ$ being perpendicular to $AB$.

By symmetry components parallel to the rod cancel.

Hence total attraction is sum of components perpendicular to rod

$$=\frac{2\gamma MN}{l^2h}\int_0^{l/2}\{\sin(2\alpha-\theta)-\sin\theta\}\,dx$$

$$=\frac{2\gamma MN}{l^2h}\int_0^{l/2}\left[\frac{l/2+x}{\{h^2+(l/2+x)^2\}^{1/2}}-\frac{l/2-x}{\{h^2+(l/2-x)^2\}^{1/2}}\right]dx$$

$$=\frac{2\gamma MN}{l^2}\{(1+l^2/h^2)^{1/2}-1\}$$

**Comment.** In this type of problem it is usually easier to obtain the attraction directly by integration, rather than from a potential function.

---

### Problem 34

Find the gravitational attraction at any point due to a uniform straight rod.

Show that if a uniform straight rod $AB$ is free to slide in a smooth groove parallel to a similar fixed rod $CD$ its time of small oscillations about its equilibrium position is $2\pi/p$, where

$$p^2 = \frac{2\gamma m}{CD}\left(\frac{1}{BD} - \frac{1}{AD}\right)$$

where $m$ is the mass of unit length of the rod and $AD$, $BD$ are distances in the equilibrium position. (O.)

---

**Solution.**

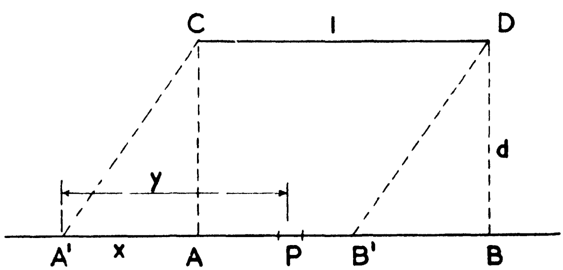


Fig. 20

Let $l$ be the length of each rod and $d$ the distance $BD$ in the equilibrium position. Consider rod $AB$ at $A'B'$, where $AA' = x$.

Clearly we are concerned only with the component of attraction parallel to $CD$, and this, on an element of length $\delta y$ at $P$, where $A'P = y$, is, by the standard bookwork (2.2 (7))

$$\gamma m^2 \left(\frac{1}{PC} - \frac{1}{PD}\right) \delta y$$

$$PC^2 = d^2 + (y - x)^2$$

$$PD^2 = d^2 + (l + x - y)^2$$

and hence the total attraction on $A^1B^1$ parallel to $CD$ is

$$\gamma m^2 \int_0^l [\{d^2 + (y - x)^2\}^{-1/2} - \{d^2 + (l + x - y)^2\}^{-1/2}]\, dy$$

Using the standard form for these integrals (or substituting $y - x = u$ in the first and $l + x - y = u$ in the second), this becomes

$$\gamma m^2 [\log [(l - x) + \{(l - x)^2 + d^2\}^{1/2}] - \log (-x + (d^2 + x^2)^{1/2}\}$$
$$- \log [(l + x) + \{(l + x)^2 + d^2\}^{1/2}] + \log \{x + (d^2 + x^2)^{1/2}\}]$$

$$\doteq \gamma m^2 \left[\log\left\{l - x + (l^2 + d^2)^{1/2} \left(1 - \frac{lx}{d^2 + l^2}\right)\right\}\right.$$
$$+ \log\{(1 + x/d)/(1 - x/d)\}$$
$$\left. - \log\left\{l + x + (l^2 + d^2)^{1/2}\left(1 + \frac{lx}{d^2 + l^2}\right)\right\}\right]$$

where we have neglected $x^2$ and higher powers on expanding the square roots, since we are considering only a small displacement of $AB$.

After expansion of the logarithms, retaining only terms in $x$, this reduces to

$$\gamma m^2\{2x/d - 2x(d^2 + l^2)^{-1/2}\}$$
$$= 2\gamma m^2 \left(\frac{1}{BD} - \frac{1}{AD}\right) x$$

leading at once to the result required.

---

**Problem 35**

Show that the potential energy of a volume distribution of gravitating matter of density $\rho$ is $\frac{1}{2}\iiint \rho\phi \, d\tau$, where $\phi$ is the gravitational potential.

A spherical shell of gravitating matter of internal and external radii $a$ and $b$ is such that its density is inversely proportional to the distance from the centre. Show that the potential energy of the shell is

$$\tfrac{2}{3}\gamma M^2(2a + b)/(a + b)^2$$

where $\gamma$ is the gravitational constant, and $M$ is the mass of the shell.

(D.)

---

**Solution.** Let the density at any point, distant $r$ from the centre be $k/r$. Then the mass of the shell is given by

$$M = \int_a^b 4\pi r^2(k/r)dr = 2\pi k(b^2 - a^2)$$

We now find the potential $\phi$ at an internal point $P$, distant $x$ from the centre, $(a \leqslant x \leqslant b)$. Considering an elemental shell bounded by spheres of radii $r$ and $r + dr$, we have, for the potential at $P$ due to such a shell, from 2.2 (9),

$$\gamma 4\pi r^2(k/r)dr/x \qquad r \leqslant x$$

or

$$\gamma 4\pi r^2(k/r)dr/r \qquad r \geqslant x$$

Thus the total potential at $P$ is

$$\phi = \frac{4\pi k\gamma}{x}\int_a^x r\,dr + 4\pi k\gamma\int_x^b dr$$
$$= \frac{2\pi k\gamma}{x}(x^2 - a^2) + 4\pi k\gamma(b - x)$$

Using the bookwork of the first part gives, for the potential energy,

$$-\pi k\gamma\int_a^b \frac{k}{x}\{(x^2 - a^2)/x + 2(b - x)\}\,4\pi x^2\,dx$$
$$= -\frac{8\pi^2k^2\gamma}{3}(b^3 - 3a^2b + 2a^3)$$
$$= -\tfrac{2}{3}\gamma M^2\frac{(b + 2a)}{(b + a)^2}$$

---

*Alternatively*, we might consider the shell as built up by bringing from infinity successive layers of mass $4\pi\rho x^2\,dx$, i.e. $4\pi kx\,dx$. Since the potential at the point $P$ due to the part of the shell already assembled is

$$2\pi k\gamma(x^2 - a^2)/x$$

the work done by the forces of attraction in adding the next layer is

$$\{2\pi k\gamma(x^2 - a^2)/x\}4\pi kx\,dx$$

Integrating this from $x = a$ to $x = b$ gives the result as before. Note that in this method we have not needed the potential at an internal point of the shell.

---

**Problem 36**

The attraction at any point inside a sphere of radius $a$ is $\dfrac{4}{3a}\pi\gamma\rho r^2$. Find the mean density of the sphere and show that the loss of gravitational potential energy in assembling the particles from a state of diffusion at an infinite distance is $\dfrac{4}{7a}\gamma M^2$, where $M$ is the mass of the sphere. (L.)

---

**Solution.** If $N$ is the attraction at the surface and $M_r$ the mass of a sphere of radius $r$, then, by Gauss's theorem,

$$-4\pi r^2N = -4\pi\gamma M_r$$

Here
$$N = \frac{4\pi\gamma\rho r^2}{3a}$$

whence
$$M_r = \frac{4\pi\rho r^4}{3a}$$

Thus when $$r = a, \; M = \frac{4\pi a^3\rho}{3}$$

and the mean density is $\rho$.

---

Let the density at a point distant $x$ from the centre be $f(x)$. Then the mass of a sphere of radius $r$ is

$$M_r = \int_0^r 4\pi f(x)\, x^2\, dx = \frac{4\pi\rho r^4}{3a} \text{ (from the above working)}$$

Differentiating w.r.t. $r$ gives

$$f(r)r^2 = \frac{4\rho r^3}{3a}$$

$$\therefore \qquad f(r) = \frac{4\rho r}{3a}$$

If we now consider the sphere to be assembled by bringing up from infinity successive spherical layers of thickness $\delta r$ we have, for the potential at the surface when the radius is $r$,

$$\frac{\gamma M_r}{r} = \frac{\gamma 4\pi\rho r^3}{3a}$$

and hence the loss of gravitational potential energy

$$= \int_0^a \frac{\gamma 4\pi\rho r^3}{3a} \cdot \frac{4\rho r}{3a} \cdot 4\pi r^2\, dr = \frac{4M^2\gamma}{7a}$$

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**Problem 37**

The potential of a gravitational field at the point $(x, y, z)$, distance $r$ from the origin, is given by

$$V_1 = \gamma k(13a^2 - 3x^2) \qquad\qquad \text{for } r \leqslant a$$
$$V_2 = \gamma k a^3(12r^4 + a^2r^2 - 3a^2x^2)/r^5 \text{ for } r > a$$

where $k$ is a positive constant and $\gamma$ the gravitational constant.

Show that the potential arises from a sphere of uniform density having radius $a$, together with a surface distribution of density $15k(a^2 - x^2)/4\pi a$ on its surface. (R.)

---

**Solution.**

$$\nabla^2 V_1 = \frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2} = -6\gamma k$$

Hence, by Poisson's equation (2.2 (4)), for $r \leqslant a$ there is a uniform volume density, magnitude $3k/2\pi$.

Using $r^2 = x^2 + y^2 + z^2$, and hence $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$, etc., we readily find

$$\frac{\partial V_2}{\partial x} = \gamma k a^3 \left( -\frac{12}{r^2} \cdot \frac{x}{r} - \frac{3a^2}{r^4} \cdot \frac{x}{r} - \frac{6a^2 x}{r^5} + \frac{15a^2x^2}{r^6} \cdot \frac{x}{r} \right)$$

$$= \gamma k a^3 \left( -\frac{12x}{r^3} - \frac{9a^2x}{r^5} + \frac{15a^2x^3}{r^7} \right)$$

and so $$\frac{\partial^2 V_2}{\partial x^2} = \gamma k a^3 \left( -\frac{12}{r^3} + \frac{36x^2}{r^5} - \frac{9a^2}{r^5} + \frac{45a^2x^2}{r^7} + \frac{45a^2x^2}{r^7} - \frac{105a^2x^4}{r^9} \right)$$

Combining this with similar results for $\dfrac{\partial^2 V_2}{\partial y^2}$ and $\dfrac{\partial^2 V_2}{\partial z^2}$ (remembering that $r^2 = x^2 + y^2 + z^2$) gives $\nabla^2 V_2 = 0$.

Hence for $r > a$ there is no distribution of matter.

To find the distribution on the surface of the sphere $r = a$ we use the result

$$\left(\frac{\partial V_2}{\partial r}\right)_{r=a} - \left(\frac{\partial V_1}{\partial r}\right)_{r=a} = -4\pi\gamma\sigma \text{ [2.2 (3) and (5)]}$$

$\sigma$ being the surface density

$$\frac{\partial x}{\partial r} = \frac{x}{r}$$

(as we see by writing $x = r \sin \theta \cos \omega$, in spherical polars) and hence

$$\frac{\partial V_1}{\partial r} = -\frac{6\gamma k x^2}{r}$$

and

$$\frac{\partial V_2}{\partial r} = \gamma k a^3 \left( -\frac{12}{r^2} - \frac{3a^2}{r^4} + \frac{9a^2x^2}{r^6} \right)$$

Putting $r = a$, we get

$$\sigma = \frac{15k}{4\pi} (a - x^2/a)$$

as required.

---

## PROBLEMS FOR SOLUTION

1. Point charges $e_1, e_2, \ldots e_n$ are at the points $X_1, X_2, \ldots X_n$ of the straight line $OX$. Prove that along a line of force,

$$\sum_{m=1}^{n} e_m \cos \theta_m = \text{constant}$$

where, if $P$ is any point on the line of force, $P\widehat{X_m}X = \theta_m$.

In the particular case when there are two charges only, namely $4e$ at $X_1$ and $-e$ at $X_2$, prove that the line of force leaving $X_1$ at an angle $\pi/3$ with $X_1X_2$ has equation $\sin \frac{1}{2}\theta_1 = \frac{1}{2} \sin \frac{1}{2}\theta_2$.

Hence or otherwise prove this line of force passes through the neutral point.

(H.)

2. If the equation $f(x, y, z) = \lambda$, where $\lambda$ is a parameter, represents a system of equipotential surfaces in a region devoid of matter, prove that

$$\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) \Big/ \left\{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right\}$$

is a function of $\lambda$ only.

Show that $(x - c)^2 + y^2 = \lambda\{(x + c)^2 + y^2\}$ represents a family of equipotential surfaces and that

$$V = A \log \lambda + B$$

where $V$ is the potential function and $A$ and $B$ are arbitrary constants. (L.)

3. Define the coefficients of capacity and potential of a system of conductors. Show that they form symmetrical matrices, assuming the conductors are embedded in an infinite dielectric.

Three equal conducting spheres are placed with their centres at the vertices of an equilateral triangle and are initially isolated and uncharged. Conductors 1, 2, and 3 are in turn charged to potential $V$ and then isolated. If the charges on 1 and 2 are then $Q_1$ and $Q_2$, find the charge on 3. (M.)

4. Define the coefficients of potential, capacity, and induction for a system of $n$ conductors.

Show that the coefficients of potential $p_{rs}$ are all positive and that $p_{rr} \geqslant p_{rs}$ $(r \neq s)$.

Obtain all the coefficients for three concentric spherical conducting, thin shells of radii $a$, $2a$, and $3a$, the intervening dielectric being air. (E.)

5. Three conductors $A_1$, $A_2$, and $A_3$ are such that $A_2$ is inside $A_3$. Show that there are four independent coefficients of potential.

$A_1$ is given a charge $Q$, and initially $A_2$ and $A_3$ are uncharged. $A_1$ is now connected to $A_2$ by a fine wire passing through a small hole in $A_3$; contact is broken and $A_1$ is now connected to $A_3$. The process of alternately connecting $A_1$ with $A_2$ and $A_3$ is repeated. If $R_r$ and $S_r$ are the charges on $A_2$ and $A_3$ after they have been connected $r$ times with $A_1$, write down the equation between the charges when $A_1$ is connected to $A_3$ for the $r$th time. Write down the corresponding equation when $A_1$ is connected to $A_2$ for the $(r + 1)$th time. Hence or otherwise show that

$$R_{r+1} = \frac{Q\alpha}{(\alpha + \gamma)} \left(\frac{\alpha + \beta}{\alpha + \gamma}\right)^r$$

where $\alpha = p_{11} - p_{12}$, $\beta = p_{33} - p_{12}$, $\gamma = p_{22} - p_{12}$.

Find also the charge on $A_1$ after it has been connected to $A_3$ for the $(r + 1)$th time. (D.)

6. Find the equations to the equipotential surfaces and lines of force of the electrostatic field due to charges $9e$ and $-e$ at distance $10a$ apart.

Show that one of the equipotential surfaces is a sphere of radius $9a/8$ and examine in particular the surface which passes through the point of equilibrium.

Sketch the equipotential surfaces and lines of force. (E.)

7. Charges $e$, $-2e$, $8e$ are placed at the points $(-a, 0, 0)$, $(0, 0, 0)$ and $(a, 0, 0)$ respectively. Prove that the neutral points in the plane $z = 0$ are

$$\left(-\frac{a}{2}, \pm\frac{a}{2}\sqrt{\frac{5}{3}}, 0\right).$$

Hence show that the extreme line of force passing from the charge $e$ to the charge $-2e$ makes on leaving $e$ an angle $\alpha$ with the $x$-axis where

$$\cos \alpha = 6 - \frac{9}{2}\sqrt{\frac{3}{2}} \qquad \text{(E.)}$$

8. Two concentric spherical surfaces, radii $a$, $b$ ($a > b$) insulated from each other and from earth, are given charges $e_1$ and $e_2$. After the spheres have been charged the inner sphere is connected to an infinitely distant uncharged insulated sphere of radius $c$. Find the charge $Q$ which this sphere receives.

If the two concentric spheres are now connected to each other and the charge on the distant sphere becomes $-2Q$, prove that

$$\frac{e_1}{e_2} = -\frac{a(2a + b + 3c)}{3ab + 2bc + ac}$$

Find the electrostatic energy of the system in its final state. (L.)

9. Prove that a conductor at potential $V$ and carrying charge $Q$ has potential energy $\frac{1}{2}QV$.

Three concentric insulated spherical conductors have radii $a$, $b$, $c$ ($a < b < c$) and carry charges $e_1$, $e_2$, $e_3$ respectively. If the three conductors are joined together by fine wires, show that there will be a loss of energy and find this loss. (D.)

10. An equipotential surface $S$, whose potential is $V$, completely surrounds a system of charges whose algebraic sum is $Q$. Explain why, if $S$ is replaced by a thin conducting shell the electric field is unaltered. Show that the capacity of an isolated condenser, consisting of the surface $S$ and the sphere at infinity, is $Q/V$.

Two such equipotential surfaces $S_1$, $S_2$, remote from each other, are at potentials $V_1$, $V_2$ and enclose charges whose algebraic sums are $Q_1$, $Q_2$, respectively. Each is now replaced by a thin conducting shell. If these shells are now joined by a thin conducting wire, prove that they attain a potential

$$\frac{V_1V_2(Q_1 + Q_2)}{V_1Q_2 + V_2Q_1}$$

Find the energy dissipated when the connection is made. (D.)

11. A parallel-plate condenser has plates of area $A$ at a distance $t$ apart. Ignoring end effects, find the capacity of the condenser and show that it has electrostatic energy $AV^2/8\pi t$ when charged to a potential difference $V$. Find also the force of attraction $F$ between the plates.

If the plates of the condenser move farther apart through a small distance $at$, the potential difference $V$ being maintained constant by connecting the plates to a suitable battery, find: (i) the loss in electrostatic energy of the condenser; (ii) the work done by the force $F$; (iii) the energy supplied to the battery, and hence check the energy balance. (C.)

12. Three concentric conducting spheres have radii $a$, $2a$, and $3a$, and carry charges $e$, 0, and $-2e$ respectively. They are *in vacuo* except for the region between the middle and outer spheres, which is filled with a substance of dielectric constant $K$. Show that the spheres have potentials in the ratio $-(K + 1) : 2K - 1 : 2K$. If the middle sphere is now earthed, prove that the electrostatic energy of the system will decrease by an amount $(2K - 1)^2e^2/6K(2K + 1)a$. (H.)

13. State Gauss's Theorem of electrostatic induction.

An insulated spherical conducting shell of inner radius $b$ and outer radius $c$ carrying a charge $Q$ is concentric with an inner earthed conducting sphere of radius $a(<b)$. The dielectric constant of the region $c < r$ is $K$, and in $a < r < b$ the dielectric constant is $K\left(\frac{a}{r}\right)^2$, where $r$ is the radial distance from the centre of the system. The shell is divided into two segments by a diametral plane $P$, and these are maintained in electrical contact by a smooth, inextensible string just fitting the outer surface of the shell and lying along a great circle in a section perpendicular to the string is in tension, show that the tension $T$ is given by

$$T = \frac{Q^2\{b^2(b - a)^2 - a^4\}}{16b^2\{c(b - a) + a^2\}^2}$$

with the condition $b > (\sqrt{5} + 1)a/2$. (N.)

14. State the equations satisfied by the electrostatic field in a region where there is a volume charge density $\rho$ and no dielectric material.

Assuming that the electric charge $Ze$ of an atomic nucleus is uniformly distributed inside a sphere of radius $R$, prove that the potential at a distance $r$ from the centre is

$$\phi = \frac{Ze}{2R}\left\{3 - \left(\frac{r}{R}\right)^2\right\}, \; r \leqslant R$$

Obtain an expression for the electrostatic energy $\frac{1}{2}\int \rho\phi d\tau$ of the nucleus and verify that it is equal to the field energy $\frac{1}{8\pi}\int E^2 d\tau$. (C.)

15. Find the radial and transverse components of the electric field due to an electric dipole of moment $m$.

Electric dipoles of moments $\sqrt{3}m$, $2m$, $m$, and $m$ are fixed at the vertices $A$, $B$, $D$, and $F$ respectively of a regular hexagon $ABCDEF$. The four dipoles are fixed in direction and point in the direction $BA$. Another dipole has its centre fixed at $E$ and is free to rotate about its centre in the plane of the hexagon. Show that it will rest with its axis making angle $\alpha$ with $DE$, where $\tan \alpha = 27\sqrt{3}/65$. (D.)

16. Show that an electric dipole of moment $\mathbf{M}$ placed at a distance $\mathbf{r}$ from a point charge $e$ experiences a force $\frac{e\mathbf{M}}{r^3} - \frac{3e(\mathbf{M}\,.\,\mathbf{r})\mathbf{r}}{r^5}$ and a couple $\frac{e\mathbf{M} \times \mathbf{r}}{r^3}$.

At the points $(d, 0, 0)$ $(-d, 0, 0)$ of a system of rectangular Cartesian axes are placed charges $e$, $-e$ respectively and a dipole is placed at the point $(0, 0, l)$. Show that the dipole experiences the same force as it would do in the presence of a dipole of moment

$$2ed\left(l + \frac{d^2}{l^2}\right)^{-5/2}$$

at the origin, pointing along the positive $x$ — axis. (D.)

17. Obtain the magnetic field intensity $\mathbf{H}$ due to a small magnet of moment $\mathbf{M}_1$ and deduce the potential energy of a second small magnet $\mathbf{M}_2$ in the field of the first.

A small magnet of moment $\mathbf{M}$ is fixed in space and another of moment $\mathbf{m}$ is free to rotate about its centre, which is on the axis of the first magnet and at a distance $b$ from it. Show that if $I$ is the moment of inertia of the free magnet about its centre the period of its small oscillations in a plane containing the magnet $\mathbf{M}$ is

$$(2\pi^2 bI/mM)^{1/2} \qquad \text{(E.)}$$

18. Two small magnets, moments $M_1$, $M_2$ are fixed with their centres at the corners $B$, $C$ of an equilateral triangle $ABC$ and with their axes pointing in the directions $BA$, $AC$. A third small magnet, $M_3$, is free to turn in the plane of the triangle about its centre, which is at $A$. Show that the axis of $M$ is inclined to the bisector of $B\hat{A}C$ at $\tan^{-1} \frac{M_1 + M_2}{\sqrt{3}(M_1 - M_2)}$ in the positions of equilibrium. If $M_1 > M_2$, which of these positions is stable? Determine the period of small oscillations about the position of stable equilibrium given that the magnet has moment of inertia $I$ about its centre. (D.)

19. Prove that the potential energy of a magnetic doublet of strength $\mathbf{m}$ in a field of magnetic intensity $\mathbf{H}$ is $-\mathbf{m}\,.\,\mathbf{H}$ and that the mutual potential energy of two doublets $\mathbf{m}$, $\mathbf{m}^1$, whose separation is $\mathbf{r}$ is

$$W = \frac{\mathbf{m}\,.\,\mathbf{m}^1}{r^3} - \frac{3(\mathbf{m}\,.\,\mathbf{r})(\mathbf{m}^1\,.\,\mathbf{r})}{r^5}$$

Two small coplanar magnets of moments $2m$, $3m$ are free to turn about their centres, which are fixed and are placed with their line of centre perpendicular to a uniform field $H$. Calculate the potential energy of the system and show that the

position of equilibrium in which their axes are in the direction of $H$ is stable if $8m < Hd^3$, where $d$ is the distance between the centres. (L.)

20. Prove that the mutual potential energy of two magnetic dipoles, of moments $\mathbf{m}, \mathbf{m}^1$ at a distance $r$ apart, is

$$\{\mathbf{m} \cdot \mathbf{m}^1 - 3(\mathbf{m} \cdot \mathbf{a})(\mathbf{m}^1 \cdot \mathbf{a})\}/r^3$$

where $\mathbf{a}$ is a unit vector along the line joining the dipoles.

Two small magnets of equal moment $m$ are free to turn about vertical axes through their centres, which are at a distance $r$ apart. The line joining their centres is horizontal and perpendicular to the earth's magnetic field. If $H$ is the horizontal component of the latter field and $H < 3m/r^3$, show that there is a position of stable equilibrium with the axes of the magnets parallel and inclined to the line of centres at $\sin^{-1}(Hr^3/3m)$. (D.)

21. Show that the scalar potential at a point $P$ due to a uniform normally magnetised shell of strength $I$ is $I\omega$, where $\omega$ is the solid angle subtended at $P$ by the shell.

A sphere of radius $a$ is magnetised so that the intensity of magnetisation at a point whose position vector is $\mathbf{r}$ referred to the centre of the sphere is $f(r)\mathbf{r} + \mathbf{c}$, where $f(r)$ is any function of $r$ $(= |\mathbf{r}|)$ and $\mathbf{c}$ is a constant vector. Prove that the field at an external point is the same as if the sphere were uniformly magnetised to the intensity $\mathbf{c}$.

Show also that the potential at an internal point is

$$4\pi \int_r^a rf(r)dr + \tfrac{4}{3}\pi\,\mathbf{r} \cdot \mathbf{c} \qquad \text{(L.)}$$

22. Prove that the potential energy of a magnetic shell of any form and of uniform strength $I$, placed in a magnetic field of intensity $\mathbf{H}$, is equal to

$$-I \iint \mathbf{H} \cdot d\mathbf{S}$$

the integral extending over the positive face of the shell. Deduce that the total force on the shell is given by

$$-I \int \mathbf{H} \times d\mathbf{s}$$

where the integral is taken round the boundary of the shell.

A small magnet of moment $m$ lies along the axis of a flat circular uniform magnetic shell of strength $I$ and radius $a$, its distance from the shell being $b$. Find the mutual force between the magnet and the shell. (D.)

23. State the equations which determine the steady flow of current in an isotropic conductor and indicate their physical meaning.

$A$ and $B$ are opposite ends of the diameter of a thin spherical shell of centre $O$, radius $a$ and thickness $t$. Current enters and leaves by two small circular electrodes of radius $c$ $(\gg t)$, whose centres are at $A$ and $B$. If $i$ is the total current and $P$ is a point on the shell such that $P\hat{O}A = \theta$, show that the magnitude of the current vector at $P$ is $i/2\pi at \sin\theta$. Deduce that the resistance of the conductor is approximately $\dfrac{1}{\pi\sigma t}\log\dfrac{2a}{c}$ where $\sigma$ is the conductivity. (C.)

24. Obtain the following equations for steady current flow in a non-uniform medium of resistivity $\tau$ and specific inductive capacity $K$, in which $\rho$, $V$, and $\mathbf{c}$ are respectively the electric volume density, the electric potential, and the current intensity

$$\operatorname{div} \mathbf{c} = 0 \qquad \nabla^2 V = \frac{\partial(\log\tau)}{\partial\mathbf{r}} \cdot \frac{\partial V}{\partial \mathbf{r}}$$

$$K\nabla^2 V + \frac{\partial K}{\partial \mathbf{r}} \cdot \frac{\partial V}{\partial \mathbf{r}} = -4\pi\rho$$

A perfectly conducting sphere, of radius $a$, is surrounded by two concentric spherical shells of uniform material; the inner and outer radii of these shells are $a$ and $b$, and $b$ and $c$, and their resistivities are $\tau$ and $2\tau$ respectively. The rest of the space is filled in by a uniform medium of resistivity $3\tau$. Show that the rate of outflow of current from the conducting sphere, when it is maintained at potential $V_0$ relative to points at an infinite distance, is given by $4\pi abcV_0/\tau\,(ab + bc + ca)$. (S.)

25. Show that the gravitational attraction due to an infinite rod of mass $m$ per unit length at a point, perpendicular distance $r$ from the rod, is $2\gamma m/r$ in the direction of the perpendicular from the point to the rod, where $\gamma$ is the gravitational constant.

An infinite rod of mass $m$ per unit length lies along the $x$-axis. A finite rod of length $a$ and mass $m$ per unit length lies parallel to the $xy$ plane with its ends at the points $(0, 0, h)$ and $(a \cos \alpha, a \sin \alpha, h)$ $(\alpha \neq 0)$. Show that the $z$-component of the gravitational force between the two rods is $2\gamma m^2\beta/\sin \alpha$ where $\beta = \tan^{-1} (a \sin \alpha/h)$ and find the $y$-component, also in terms of $\beta$. (D.)

26. Prove that the attraction of a uniform circular disc of radius $a$ and surface density $\sigma$ at a point on the axis distant $p$ from the disc is

$$2\pi\gamma\sigma\{1 - p(a^2 + p^2)^{-\frac{1}{2}}\}$$

Prove that the attraction of a uniform solid hemisphere at the point on its curved surface farthest from the base is $2(3 - \sqrt{2})/3$ times the attraction at the centre of the base. (R.)

27. Find the potential of a thin uniform shell of matter at: (i) an internal point; (ii) an external point.

The density in a sphere of matter is $\rho_0 - \lambda r^2$ at a distance $r$ from the centre, where $\rho_0$ is the density at the centre and $\lambda$ is constant. Find the potential at any internal point, and prove that the potential at the centre is

$$\pi Ga^2(\rho_0 + \rho_1)$$

where $a$ is the radius of the sphere, $\rho_1$ the density at its surface, and $G$ the gravitational constant. (D.)

28. A spherical shell of gravitating matter of uniform density $\rho$ has internal and external radii $b$ and $c$ respectively. Show that the work done in dissipating the shell completely to infinity is (in gravitational units)

$$\tfrac{8}{15}\,\pi^2\rho^2(2c^5 - 5c^2b^3 + 3b^5)$$

Find the potential $\phi$ at any point of the complete shell and verify that the above expression is equal to

$$\tfrac{1}{2}\int \rho\phi\, d\tau$$

where the integration is over the complete shell. (D.)

29. Show that the potential energy of a volume distribution of gravitating matter of density $\rho$ is $-\frac{1}{2}\iiint_V \rho V dv$, where $V$ is the gravitational potential. The density at a point inside a solid sphere of mass $M$ and radius $a$ is proportional to the square of the distance from the centre. Show that the potential energy of the sphere is $-5\gamma M^2/9a$, where $\gamma$ is the gravitational constant. (D.)

30. Assuming the inverse-square law of attraction, obtain Poisson's equation $\nabla^2\phi = -4\pi G\rho$ for the gravitational potential $\phi$ in a continuous distribution of material of variable density $\rho$, $G$ being the gravitational constant.

A small portion of a continuous gravitating medium is bounded by a sphere of radius $a$. Show that the force on it per unit volume due to the action of the remainder of the material is given approximately by the vector expression

$$\rho \operatorname{grad} \phi + \tfrac{1}{10}\, a^2\nabla^2(\rho \operatorname{grad} \phi)$$

evaluated at the centre of the sphere. (L.)

31. If $V_1$, $V_2$ denote the potentials at points $P_1$ and $P_2$ on opposite sides of a surface distribution of matter and $dn$ denotes an element of normal taken positively from $P_1$ to $P_2$, prove that the surface density is

$$\frac{1}{4\pi}\left(\frac{\partial V_1}{\partial n} - \frac{\partial V_2}{\partial n}\right)$$

evaluated at the surface.

The potential of a certain distribution of matter at a point $(x, y, z)$ is

$$\frac{4}{3}\gamma\sigma\pi a^4/r + \frac{4}{15}\gamma\sigma\pi a^6\left(\frac{3x^2}{r^5} - \frac{1}{r^3}\right) \qquad r > a$$

and

$$\tfrac{4}{3}\gamma\sigma\pi a^3 + \tfrac{4}{15}\gamma\sigma\pi a(3x^2 - r^2) \qquad r < a$$

Find the distribution of matter. (L.)

## CHAPTER 3

# HYDRODYNAMICS

In this chapter some simpler problems of non-viscous hydrodynamics are solved, dealing with uniform streams, sources and sinks, and Bernouilli's equation applied to various conditions. More specialised problems of two-dimensional motion and spherical boundaries, together with particular techniques, are covered in Vol. II.

### 3.1 Basic Equations

Non-viscous fluid, density $\rho$, velocity $\mathbf{q}$, in laminar flow, satisfies

(*a*) equation of continuity $\dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{q}) = 0$

For incompressible fluid this becomes $\operatorname{div} \mathbf{q} = 0$.
The acceleration of a fluid particle is

$$\mathbf{f} = \frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla)\mathbf{q} = \frac{\partial \mathbf{q}}{\partial t} + \operatorname{grad}(\tfrac{1}{2}\mathbf{q}^2) - \mathbf{q} \times \operatorname{curl} \mathbf{q}$$

and the

(*b*) equation of motion is $\mathbf{f} = \mathbf{F} - \dfrac{1}{\rho} \operatorname{grad} p$

where $\mathbf{F}$ is the body force per unit mass and $p$ is the fluid pressure, which, for a non-viscous fluid, is the same in all directions (and so is a scalar quantity).

**3.1.1 Bernouilli's Equation.** If the body forces are conservative, and the pressure is a function of the density, then $\mathbf{F} = -\operatorname{grad} V$, and

(*a*) in *steady flow* $\frac{1}{2}q^2 + V + \int dp/\rho$ is constant along a streamline;

(*b*) in *irrotational motion* (see 3.2) $-\dfrac{\partial \phi}{\partial t} + \frac{1}{2}q^2 + V + \int dp/\rho = G(t)$,

a function of time only;

(*c*) in *irrotational steady motion* $\frac{1}{2}q^2 + V + \int dp/\rho$ is constant everywhere.

**3.1.2 Boundary Conditions.** At any boundary the normal velocities of fluid and boundary are equal.

### 3.2 Irrotational Motion

This is defined by curl $\mathbf{q} = 0$, and so $\mathbf{q} = -\text{grad}\ \phi$, $\phi$ being the potential (single valued in a simply connected region).

For an incompressible fluid, the equation of continuity then gives

$$\nabla^2\phi = 0$$

**3.2.1 Uniqueness Theorems.** There cannot be two motions satisfying $\nabla^2\phi = 0$ and prescribed motions of boundaries.

**3.2.2 Some Potential Functions in Three Dimensions.** For a source of strength $m$ at the origin, $\phi = m/r$, where $m$ is defined as $\frac{1}{4\pi} \times$ (rate of emission of volume).

For a doublet $M$ at the origin, $\phi = M \cos \theta/r^2$.

For a uniform stream in positive $x$ direction, $\phi = -Ux$.

### 3.3 Impulsive Motion

The general equation of impulsive motion is

$$\mathbf{q}_1 - \mathbf{q}_0 = \mathbf{I} - \frac{1}{\rho}\ \text{grad}\ p$$

where $\mathbf{I}$ is the impulse and $p$ the impulsive pressure. (If the impulse is produced solely by the motion of the boundaries, $\mathbf{I} = 0$.)

**3.3.1** A motion generated from rest by impulsive pressure from boundaries only is irrotational, and the impulsive pressure is given by

$$p = \rho\phi \text{ everywhere.}$$

### 3.4 Kinetic Energy

The kinetic energy of a fluid is $\frac{1}{2}\rho\int_v q^2\,dv$, which for irrotational motion reduces (by Green's Theorem) to

$$\text{K.E.} = \tfrac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n}\,dS$$

$\frac{\partial}{\partial n}$ meaning differentiation along the outward normal, as usual.

### 3.5 Dynamical Principles

The ordinary laws of dynamics hold, and may often be used to shorten the solution of a problem; in particular,

(*a*) the momentum produced by an impulse is equal to the impulse;
(*b*) the kinetic energy produced by an impulse is the product of the

impulse, and the mean of the initial and final velocities of its point of application in the direction of the impulse;

(*c*) the rate of increase of energy in a system is the rate at which work is done on the system.

---

**Problem 38**

An infinite mass of inviscid liquid of constant density $\rho$ is initially at rest, and has a spherical cavity of radius $a$. The liquid is made to move outwards by a pressure applied uniformly over the surface of the cavity; there is no pressure at infinity and no body forces act. If the radius of the cavity is $b$ at time $t$, and the pressure applied is $Kb^{-3}$, $K$ constant, prove that

$$\rho b^3\left(\frac{db}{dt}\right)^2 = 2K \log (b/a)$$

and that the pressure at distance $r$ from the centre of the sphere is

$$(K/b^2r)[1 + \{1 - b^3/r^3\} \log b/a] \qquad \text{(L.)}$$

---

**Solution.** The motion is purely radial, and hence must be irrotational, with potential

$$\phi = A/r \quad \text{(where } A \text{ is independent of } r\text{)}$$

To find $A$, in terms of the other quantities, we solve for the outward velocity at the bubble boundary, which is

$$\frac{db}{dt} \text{ and is also } \frac{A}{b^2}$$

Hence
$$A = b^2\frac{db}{dt} \text{ and } \phi = \frac{b^2}{r}\left(\frac{db}{dt}\right)$$

Now we can use Bernouilli's theorem (3.1.1) in the form applicable to irrotational non-steady motion; this gives, with the usual notation,

$$p/\rho - \frac{\partial\phi}{\partial t} + V + \tfrac{1}{2}q^2 = G(t) \quad . \quad . \quad . \quad . \quad (1)$$

where $G(t)$ is a function of time only, i.e. is constant throughout all space at any instant. Here $V = 0$, $q = A/r^2$, $\dfrac{\partial\phi}{\partial t} = \dfrac{1}{r}\dfrac{dA}{dt}$ and by considering conditions as $r \longrightarrow \infty$ where $p = 0$ it can be seen that $G(t) = 0$, all $t$.

Hence (1) becomes, substituting for $A$,

$$\frac{p}{\rho} - \frac{\left\{b^2\dfrac{d^2b}{dt^2} + 2b\left(\dfrac{db}{dt}\right)^2\right\}}{r} + \frac{1}{2}\frac{b^4\left(\dfrac{db}{dt}\right)^2}{r^4} = 0 \quad . \quad . \quad (2)$$

and, since $p = Kb^{-3}$ when $r = b$, this reduces to

$$K/\rho b^3 = \frac{3}{2}\left(\frac{db}{dt}\right)^2 + b\frac{d^2b}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Using $\dfrac{d^2b}{dt^2} = \frac{1}{2}\dfrac{d}{db}\left(\dfrac{db}{dt}\right)^2$, and $\dfrac{db}{dt} = 0$ when $b = a$, this integrates to

$$b^3\left(\frac{db}{dt}\right)^2 = (2K/\rho)\log b/a$$

Substituting from this in (2) gives the relationship

$$p = (K/b^2r)\,[1 + \{1 - b^3/r^3\}\log(b/a)]$$

*Alternatively* the problem may be done by using energy, which shortens the working by obtaining a first integral directly. In this we work as before up to $\phi = \dfrac{b^2}{r}\dfrac{db}{dt}$. Then the kinetic energy, at the instant when the radius of the bubble is $b$, is from 3.4,

$$\frac{1}{2}\rho\int\phi\frac{\partial\phi}{\partial n}\,dS = -\frac{1}{2}\rho\left[\phi\frac{\partial\phi}{\partial r}\,.\,4\pi r^2\right]_{r=b}$$

(since $\phi$, $\partial\phi/\partial r$ are constant over $r = b$)

$$= \frac{1}{2}\,.\,4\pi\rho b^3\left(\frac{db}{dt}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The motion starts from rest and so this energy is the work done by the pressure in expanding the bubble from initial radius $a$ to $b$.

The work done by pressure $p$ in expanding from radius $b$ to $b + db$ is $4\pi b^2p\,db$, and $p = K/b^3$, giving

$$\text{total work} = \int_a^b\frac{K}{b^3}\,.\,4\pi b^2db = 4\pi K\log b/a \quad . \quad . \quad (5)$$

Equating (1) and (2) gives the first equation required

$$\rho b^3\left(\frac{db}{dt}\right)^2 = 2K\log b/a$$

The pressure at any distance $r$ has then to be found from Bernouilli's equation as before.

---

**Problem 39**

Infinite inviscid liquid of constant density is attracted towards a fixed point $O$ by a force $f(r)$ per unit mass, $r$ being the distance from $O$. Initially the liquid is at rest, and there is a cavity bounded by a sphere

$r = a$. If there is no pressure at infinity or in the cavity, prove that the radius $R$ of the cavity at time $t$ is such that

$$\frac{d}{dt}\left\{R^3\left(\frac{dR}{dt}\right)^2\right\} + 2R^2\frac{dR}{dt}\int_R^\infty f(r)\,dr = 0$$

If $f(r) = Kr^{-3/2}$, $K$ constant, and the cavity is filled after time $T$, prove that $25\ K^2T^4 = 4a^5$. (L.)

---

**Solution.** The motion is wholly radial and so irrotational, with potential $\phi = A/r$, $A$ a constant. To find $A$ in terms of other quantities we solve for the velocity at the bubble boundary, which is

$$-\frac{dR}{dt}\text{ (inwards) and also } - A/R^2 \text{ (inwards)}$$

Thus, (as in problem 38), $A = R^2\dfrac{dR}{dt}$ and $\phi = \dfrac{R^2}{r}\dfrac{dR}{dt}$.

Now use Bernouilli's theorem (3.1.1) for irrotational non-steady motion; this gives

$$p/\rho - \frac{\partial\phi}{\partial t} + V + \tfrac{1}{2}q^2 = G(t) \quad . \quad . \quad . \quad . \quad (1)$$

where $G(t)$ is a function of time only, i.e. is constant throughout all space at any instant.

Here the potential of external forces is $V = -\int_r f(r)\,dr$, and $\dfrac{\partial\phi}{\partial t} = \dfrac{1}{r}\dfrac{dA}{dt}$, $q = A/r^2$, and by considering conditions as $r \longrightarrow \infty$ where $V = 0$ and $p = 0$, it can be seen that $G(t) = 0$, all $t$. Since also $p = 0$ on $r = R$, substituting these values and $r = R$ in (1) gives

$$R\frac{d^2R}{dt^2} + 2\left(\frac{dR}{dt}\right)^2 + \int_R^\infty f(r)\,dr - \tfrac{1}{2}\frac{R^4}{r^4}\left(\frac{dR}{dt}\right)^2 = 0$$

and this can be put into the form

$$\frac{d}{dt}\left\{R^3\left(\frac{dR}{dt}\right)^2\right\} + 2R^2\frac{dR}{dt}\int_R^\infty f(r)\,dr = 0 \quad . \quad . \quad (2)$$

If $f(r) = Kr^{-3/2}$, $\displaystyle\int_R^\infty f(r)\,dr = 2KR^{-1/2}$, and using $dR/dt = 0$ when $R = a$,

the equation (2) then integrates to

$$R^3\left(\frac{dR}{dt}\right)^2 + \frac{8}{5}KR^{5/2} = \frac{8}{5}Ka^{5/2} \quad . \quad . \quad . \quad . \quad (3)$$

If $u = R^{5/2}$, $du/dt = 5/2R^{3/2}\, dR/dt$ and, substituting in (3)

$$(du/dt)^2 = 10K(a^{5/2} - u)$$

This integrates, using $R = a$, $u = a^{5/2}$ when $t = 0$, $u = 0$ when $t = T$, to give

$$T^4 = \tfrac{4}{25}\, a^5/K^2$$

*Alternatively*, the working can be shortened by using energy to obtain a first integral directly, as in Problem 38.

The kinetic energy when the radius is $R$ is, from 3.4,

$$\tfrac{1}{2}\rho\int \phi \frac{\partial \phi}{\partial n}\, dS = \tfrac{1}{2}\rho\left[\phi \frac{\partial \phi}{\partial r} 4\pi r^2\right]_{r=R} = \tfrac{1}{2}\,.\, 4\pi\rho\, R^3\left(\frac{dR}{dt}\right)^2 \quad . \quad (4)$$

The rate of change of kinetic energy is the rate of doing work, and the only force doing work is the attraction $f(r)$ per unit mass.

The rate of flow must be the same across any concentric spherical surface; and across the inner surface it is $-4\pi R^2 dR/dt$, across any other $-4\pi r^2 dr/dt$. The total rate of doing work is then

$$\int_R^\infty 4\pi r^2 \frac{dr}{dt} f(r)\, dr = 4\pi R^2 \frac{dR}{dt}\int_R^\infty f(r)\, dr \quad . \quad . \quad . \quad (5)$$

and equating this to the rate of change of kinetic energy from (4) gives equation (2).

---

**Problem 40**

If the pressure in a non-viscous fluid is a function of the density and the external forces are derived from a potential $V$, prove that the vorticity $\mathbf{W} = \frac{1}{2}$ curl $\mathbf{q}$, ($\mathbf{q}$ being the velocity vector), will be zero everywhere at all times if it is zero everywhere at $t = 0$. Show further that when $\mathbf{W} = 0$ everywhere then

$$\int \frac{dp}{\rho} + \tfrac{1}{2}q^2 + V - \frac{\partial \phi}{\partial t}$$

is constant throughout the fluid at any instant of time.

A spherical shell with inner and outer radii $a$ and $b$, of uniform incompressible gravitating liquid is at rest and surrounds a solid sphere of radius $a$. The inner sphere is suddenly annihilated. Prove that the pressure distribution in the liquid immediately afterwards is given by

$$p = 2\pi\gamma\rho^2 \frac{(b - r)(r - a)(a + b + r)}{3r} \qquad \text{(O.)}$$

---

**Solution.** The first part is bookwork.

In the second part, consider the part of the shell between $r$ and $r + dr$; the force on this is due only to the matter within it, and this attracts as if its mass were concentrated at the centre of the shell (2.2 (9)).

Thus the force of attraction initially is

$$\frac{4}{3}\pi\rho\gamma\frac{(r^3 - a^3)}{r^2} = f(r) \text{ inwards} \quad a \leqslant r \leqslant b \quad . \quad . \quad (1)$$

The liquid is then subject to this body force and to zero external and internal pressures, and the problem is similar to Problem 39.

As then, $$V = -\int_r^b f(r)\,dr \text{ initially} \quad . \quad . \quad . \quad . \quad (2)$$

and $$p/\rho - \frac{\partial\phi}{\partial t} + V + \tfrac{1}{2}q^2 = G(t) \quad . \quad . \quad . \quad . \quad (3)$$

As in Problems 38 and 39, the motion is irrotational, potential $A/r$, and if the internal radius be $R$ at time $t$ we find $A = R^2dR/dt$ by equating velocities at $r = a$. Then

$$\frac{\partial\phi}{\partial t} = \frac{1}{r}\frac{\partial A}{\partial t} = \frac{1}{r}\left\{R^2\frac{d^2R}{dt^2} + 2R\left(\frac{dR}{dt}\right)^2\right\}$$

and so initially $R = a$, $\dfrac{dR}{dt} = 0$ and $\dfrac{\partial\phi}{\partial t} = \dfrac{a^2}{r}\left(\dfrac{d^2a}{dt^2}\right)_{t=0}$ $\quad . \quad . \quad (4)$

Also $q = 0$ everywhere, and (3) gives, taking conditions at $r = b$, $t = 0$

$$-\frac{a^2}{b}\left(\frac{d^2a}{dt^2}\right)_{t=0} = G(0) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and, taking conditions at $r = a$, $t = 0$

$$-a\left(\frac{d^2a}{dt^2}\right)_{t=0} - \int_a^b f(r)\,dr = G(0) \quad . \quad . \quad . \quad (6)$$

and at $r$, $t = 0$ we have

$$p/\rho - \frac{a^2}{r}\left(\frac{d^2a}{dt^2}\right)_{t=0} - \int_r^b f(r)\,dr = G(0) \quad . \quad . \quad . \quad (7)$$

Eliminating $\left(\dfrac{d^2a}{dt^2}\right)_{t=0}$ and $G(0)$ from (5), (6), and (7) gives

$$p/\rho - \int_r^b f(r)\,dr = -\frac{a(b-r)}{r(b-a)}\int_a^b f(r)\,dr$$

and substituting from (1) for $f(r)$ and integrating, this reduces to

$$p = 2\pi\gamma\rho^2\frac{(b-r)(r-a)(a+b+r)}{3r}$$

**Comment.** This problem is the same in principle as the two previous ones, but differs slightly in technique because of the unknown $d^2a/dt^2$ and $G(0)$. Since only the initial pressure distribution is required, it is no easier to do the problem by energy methods. If the subsequent motion were required, however, energy would be a much easier method, since otherwise conditions at *two* moving boundaries are involved.

**Problem 41**

A source of strength $m$ is situated in a stream which at large distances from the source has uniform velocity $U$ and pressure $P$. Show that this combination represents irrotational flow past a certain semi-infinite surface and find the equation of the surface. Determine the part of the surface on which the pressure exceeds $P$ and find the force exerted by the fluid on this part. (M.)

**Solution.** Taking spherical polar coordinates with the source at the origin and the base line along the positive direction of the stream, we have, from 3.2 (2), potential due to source is $m/r$, potential due to stream is $-Ur\cos\theta$, so that the total potential is $m/r - Ur\cos\theta$ . (1)

and the velocity $\mathbf{q} = -\text{grad}\,\phi$

$$= \{U\cos\theta + m/r^2, -U\sin\theta, 0\} \quad . \quad . \quad . \quad (2)$$

There is a stagnation point where

$$(U\cos\theta + m/r^2)^2 + U^2\sin^2\theta = 0$$

and the only possible solution of this is $\theta = \pi$, $r = \sqrt{m/U}$

i.e. at the point $S$, where the source and stream neutralise each other. The streamline through $S$ is the dividing streamline as shown.

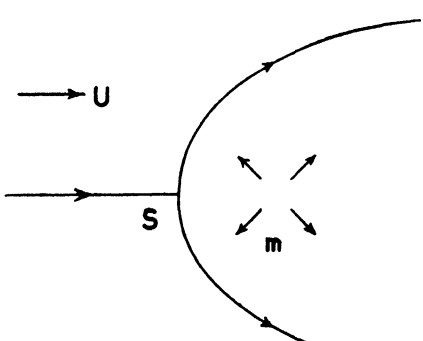


Fig. 21

The velocity at any point is tangential to the streamline, and so

$$\frac{rd\theta}{dr} = \frac{-U\sin\theta}{U\cos\theta + m/r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which integrates to $\frac{1}{2}r^2U^2\sin^2\theta - mU\cos\theta = \text{constant}$.
Thus the streamline through $S$ is

$$\tfrac{1}{2}r^2U^2\sin^2\theta - mU\cos\theta = mU$$

which simplifies to $$r = \sqrt{\frac{m}{U}}\operatorname{cosec}\theta/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This is the equation of the boundary shown of a semi-infinite solid (a solid of revolution about the base line); and since the whole boundary is a stream surface, the potential (1) represents flow up to this boundary, i.e. flow in which the shaded region contains no fluid.

Using Bernouilli's equation (3.1.1) for steady irrotational motion under no body forces, $p/\rho + \frac{1}{2}q^2$ is constant, and since at infinity $p = P$, $q = U$, we have everywhere

$$p/\rho + \tfrac{1}{2}q^2 = P/\rho + \tfrac{1}{2}U^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now from (2)

$$\begin{aligned} q^2 &= (U\cos\theta + m/r^2)^2 + U^2\sin^2\theta \\ &= U^2 + 2mU\cos\theta/r^2 + m^2/r^4 \end{aligned}$$

Hence from (5)

$$p = P + \rho\{\tfrac{1}{2}U^2 - \tfrac{1}{2}U^2 - mU\cos\theta/r^2 - \tfrac{1}{2}m^2/r^4\}$$

And on the boundary (4) this gives, after simplification,

$$p = P + \rho U^2\sin^2\theta/2\,(3/2\sin^2\theta/2 - 1) \quad . \quad . \quad . \quad (6)$$

Thus $p > P$ on the boundary as long as $\frac{3}{2}\sin^2\frac{\theta}{2} > 1$, i.e. $\sin\frac{\theta}{2} > \sqrt{2/3}$.

(the alternative $\sin\theta/2 < -\sqrt{2/3}$ is meaningless, since $0 \leqslant \theta \leqslant \pi$)

The total force exerted on the body by the stream is, by symmetry, in the direction of the stream.

The unit inward normal has components $-r\frac{d\theta}{ds}, \frac{dr}{ds}$ and the unit vector along the stream has components in the $r$, $\theta$ system ($\cos\theta$, $-\sin\theta$). Hence the thrust $2\pi r\sin\theta\, pds$ on an element of boundary has component along the stream of $-2\pi rp\sin\theta\left(r\frac{d\theta}{ds}\cos\theta + \frac{dr}{ds}\sin\theta\right)ds$. Substituting for $p$ from (6) and for $r$ from (4) this becomes, on simplifying

$$\frac{2\pi m\sin\theta}{U}\left\{P + \rho U^2\sin^2\frac{\theta}{2}\left(\frac{3}{2}\sin^2\frac{\theta}{2} - 1\right)\right\}d\theta$$

The total thrust on this part of the boundary is then

$$\frac{8\pi m}{U}\int_{\theta/2=\sin^{-1}\sqrt{2/3}}^{\theta/2=\pi/2} \sin\theta\left\{P+\rho U^2\sin^2\frac{\theta}{2}\left(\frac{3}{2}\sin^2\frac{\theta}{2}-1\right)\right\}d\left(\frac{\theta}{2}\right)$$

which gives, after integration and substitution,

$$\frac{8\pi m}{U}\left(\tfrac{1}{3}P+\tfrac{2}{27}\rho U^2\right)$$

**Comment.** The method is straightforward, but requires a considerable amount of manipulation.

**Problem 42**

Obtain the equation

$$\frac{\partial \mathbf{q}}{\partial t}+\operatorname{grad}\left(p/\rho+\tfrac{1}{2}q^2+V\right)=\mathbf{q}\times\operatorname{curl}\mathbf{q}$$

for the motion of an inviscid incompressible fluid under forces arising from a potential $V$.

A pipe of variable circular cross-section is given by $r=a(\cosh\alpha z)^{1/4}$ where $r$, $\theta$, $z$ are cylindrical polars, and the $z$-axis is vertical. Incompressible inviscid fluid is in steady irrotational motion, a volume $Q$ passing every cross-section of pipe per unit time.

Assuming the vertical velocity to be a function of $z$ only, and neglecting the horizontal velocity, show that the pressure will not be a monotonic function of $z$ if

$$\alpha Q^2 > 4\pi^2 a^4 g \qquad \text{(L.)}$$

**Solution.** In the second part the fluid flows steadily along the pipe at a volume rate of $Q$ per unit time; and since the radius is $a(\cosh\alpha z)^{1/4}$, the velocity $q$ along the pipe is given, by continuity, by

$$\pi r^2 q = Q$$

i.e.

$$q = Q(\cosh\alpha z)^{1/2}/\pi a^2 \quad . \quad . \quad . \quad . \quad (1)$$

Gravity is the only external force, and has potential $V=gz$. Thus Bernouilli's equation (3.1.1) for steady irrotational motion gives

$$p/\rho+\tfrac{1}{2}q^2+V=C$$

i.e.

$$p/\rho+\tfrac{1}{2}Q^2\cosh\alpha z/\pi^2a^4+gz=C \quad . \quad . \quad . \quad (2)$$

The pressure change along the pipe is $dp/dz$, which from (2) is given by

$$\frac{1}{\rho}\frac{dp}{dz}=-g+\frac{1}{2}\frac{Q^2\alpha}{\pi^2a^4}\frac{\sinh\alpha z}{\cosh^2\alpha z} \quad . \quad . \quad . \quad . \quad (3)$$

The condition that $p$ shall not change monotonically is that the equation $dp/dz = 0$ has real roots (since then $dp/dz$ changes sign along the pipe). Putting $\dfrac{Q^2\alpha}{2\pi^2a^4} = b$, the equation $\dfrac{dp}{dz} = 0$ is, in terms of $\sinh \alpha z$,

$$g(1 + \sinh^2 \alpha z) - b \sinh \alpha z = 0$$

which has real roots if

$$b^2 > 4g^2, \text{ i.e. } b > 2g, \text{ or } \alpha Q^2 > 4\pi^2a^4g.$$

---

**Comment.** It should be noted that the expression found for $\mathbf{q}$ does not satisfy the equation of continuity $\operatorname{div} \mathbf{q} = 0$; this is, of course, because the horizontal components have been ignored as being small compared with the lengthwise component. Being small, they would not materially affect the expression for pressure.

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**Problem 43**

Show that a perfect fluid in a conservative field of force satisfies

$$\nabla V + \frac{1}{\rho} \nabla p + \mathbf{f} = 0$$

and explain the symbols.

In a large pond each particle of water traverses a horizontal circle whose centre is on a fixed vertical axis; the speed at $r$ from the axis is $\omega r$, $r < a$, and $\omega a^2/r$ for $r > a$. Given that $\omega$, $a$ are constant, find the form of the free surface. (L.)

---

**Solution.** In the second part, the motion is *not* irrotational (since $\int \mathbf{q}.d\mathbf{s}$. round a horizontal circle traversed by a particle is not zero).

We know $\mathbf{q}$ everywhere, and $V$, the potential of external force, i.e. of gravity, is $gz$. Bernouilli's theorem (3.1.1) gives only $p/\rho + \frac{1}{2}q^2 + gz$ constant along a streamline, which, since the streamlines are horizontal circles on which $p$ and $q$ are constant, gives no new information.

It is therefore necessary to go back to the basic equations of motion.

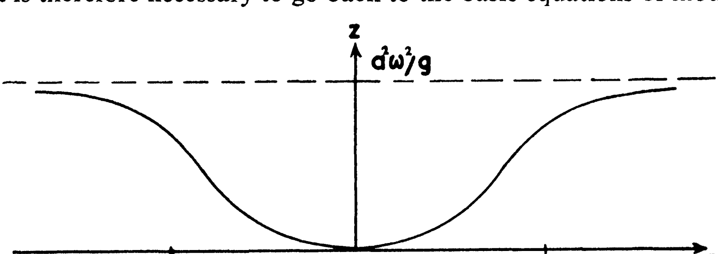


Fig. 22

Take cylindrical coordinates $r$, $\theta$, $z$ with origin at the lowest point of the free surface, and assume the pressure there to be $p_0$.

Then the acceleration $\mathbf{f}$ is due to the circular motion only, and so is

$$\begin{aligned} \mathbf{f} &= f_1, 0, 0 = -r\omega^2, 0, 0 & r < a \\ &= f_2, 0, 0 = -\omega^2 a^4/r^3, 0, 0 & r > a \\ \nabla V &= \operatorname{grad} V = 0, 0, g & \\ \nabla p &= \operatorname{grad} p = \frac{\partial p}{\partial r}, 0, \frac{\partial p}{\partial z} & \end{aligned}$$

The component equations of motion are then, from 3.1($b$),

$$\frac{1}{\rho}\frac{\partial p}{\partial r} + f = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$g + \frac{1}{\rho}\frac{\partial p}{dz} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which integrate to

$$p = p_0 - \rho g z + F(r) \text{ where } F(r) = 0, r = 0 \quad . \quad . \quad (3)$$

and

$$\begin{aligned} p &= p_0 + G(z) + \int_0^r (-f_1)\, dr \text{ for } r < a \\ &= p_0 + G(z) + \int_0^a (-f_1)\, dr + \int_a^r (-f_2)\, dr \text{ for } r > a \quad (4) \end{aligned}$$

Combining (3) and (4) and substituting for $f_1$ and $f_2$ gives finally
$p = p_0 - \rho g z + \rho r^2\omega^2/2, r < a; p = p_0 - \rho g z + \rho a^2\omega^2 - \rho\omega^2 a^4/2r^2, r > a$

From these, the free surface $p = p_0$ is

$$\begin{aligned} z &= \tfrac{1}{2}\frac{\omega^2}{g} r^2, & r < a; \\ z &= a^2\frac{\omega^2}{g} - \tfrac{1}{2}a^4\frac{\omega^2}{gr^2}, & r > a \end{aligned}$$

The surface is as sketched.

**Comment.** This is one of the few hydrodynamic examples where the original equations have to be used, since the motion is not irrotational and Bernouilli's equation, which normally provides a first integral, gives no useful information.

**Problem 44**

Prove that if the boundaries of a liquid at rest are suddenly set in motion the resulting motion of the liquid is irrotational.

A cylinder of any form of section is filled with fluid and is suddenly given an angular velocity $\omega$ about any point. Show that if the kinetic energy acquired by the fluid is $\frac{1}{2}I\omega^2$, the angular momentum acquired is $I\omega$. (C.)

**Solution.** The first part is bookwork.

The second part can be done in several ways.

(*a*) The most straightforward approach.

The motion, being started by boundary movement, is irrotational (3.3.1); suppose its potential to be $\phi$.

Then the angular momentum about $O$ is $-\rho \int_v (\mathbf{r} \times \text{grad}\, \phi)\, dv$, the integral being taken through the interior of the cylinder.

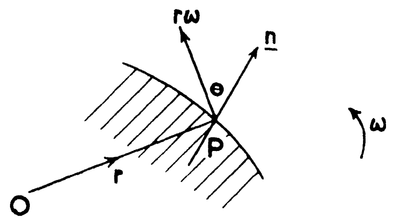


Fig. 23

To transform this to a surface integral (since it is only on the boundary that the velocity is known) we use

$$\mathbf{r} \times \text{grad}\, \phi = \text{grad}\, (\tfrac{1}{2}r^2) \times \text{grad}\, \phi = -\text{curl}\, (\phi\, \text{grad}\, \tfrac{1}{2}r^2)$$
$$\text{(since curl grad } \tfrac{1}{2}r^2 = 0)$$

$$\text{Hence } -\rho \int_a (\mathbf{r} \times \text{grad}\, \phi)\, dv = \rho \int_S \{\mathbf{n} \times \phi\, \text{grad}\, (\tfrac{1}{2}r^2)\}\, dS \text{ from 1.3 (iii)}$$

$$= \rho \int_S (\mathbf{n} \times \phi\, \mathbf{r})\, dS$$

where $\mathbf{n}$ is the normal to the surface drawn out of the liquid, and makes angle $\theta$ with the direction of the velocity at any point of the boundary.

i.e. $$\text{angular momentum} = -\rho \int_S \phi\, r \cos\theta\, dS \quad . \quad . \quad (1)$$

The kinetic energy—irrotational motion—is $\frac{1}{2}\rho \int_S \phi \frac{\partial \phi}{\partial n}\, dS$ (3.4) and, on the boundary, $\frac{\partial \phi}{\partial n}$ is the inward normal velocity $-r\omega \cos\theta$.

Hence $$\text{kinetic energy} = -\tfrac{1}{2}\rho\omega \int \phi\, r \cos\theta\, dS \quad . \quad . \quad . \quad (2)$$

Thus from (1) and (2) if the angular momentum is $I\omega$ the kinetic energy is $\frac{1}{2}I\omega^2$.

(*b*) Alternatively, the working can be shortened by using dynamical principles.

The angular momentum of the liquid must be the moment of the applied impulse, and this is simply the impulsive pressure, $\rho\phi$ on the boundary, i.e. in the direction of $-\mathbf{n}$ (3.3.1). Hence

$$\text{angular momentum} = -\int_S \rho\theta r \cos\theta \, dS$$

i.e. equation (1) is obtained directly.

(*c*) Similarly, the derivation of the kinetic energy can be shortened by using the relation that the energy is the product of the impulse and the mean of the initial and final velocities in the direction of the impulse (3.5).

Here the impulse is $\rho\phi$ along $-\mathbf{n}$, and the final velocity in this direction is $-r\omega\cos\theta$, so that immediately

$$\text{kinetic energy} = -\tfrac{1}{2}\int \rho\phi \, r\omega \cos\theta \, dS$$

giving equation (2) directly.

**Comment.** This illustrates the way in which the use of general dynamics can replace manipulation of vector fields and integrals.

**Problem 45**

Incompressible liquid of density $\rho$ flows steadily under the action of a conservative external force $-\text{grad}\ V$. If $p$ and $q$ denote the pressure and speed of the liquid at any point show that

$$p/\rho + \tfrac{1}{2}q^2 + V$$

is constant along a streamline.

Water flows steadily with a free surface under the action of gravity along a long straight channel bounded by plane vertical walls $y =$ constant. The bed of the channel is in the form of the surface $z = f(x)$ where $z$ is measured in the direction of the upward vertical and $f(x) \longrightarrow 0$ as $x \longrightarrow \pm\infty$. The gradient of the bed is everywhere small and the velocity may be taken as horizontal and uniform over any section $x =$ constant. If the depth and velocity of the water at a great distance upstream are respectively $h$ and $\sqrt{\alpha^3 gh}$, show that the depth $d$ at any $x$ satisfies

$$\frac{\alpha^3 h^3}{2d^2} + d + f(x) = (\tfrac{1}{2}\alpha^3 + 1)h$$

Hence show that the maximum possible value of $f(x)$ is

$$\tfrac{1}{2}h(\alpha - 1)^2 (\alpha + 2)$$

Show also that, if the bed of the channel attains this maximum level at one point only, the depth of water at a great distance downstream is $nh$, where $n$ is the positive root of

$$2n^2 - \alpha^3 n - \alpha^3 = 0$$

provided that there is no discontinuity in the gradient of the free surface. (C.)

---

**Solution.**

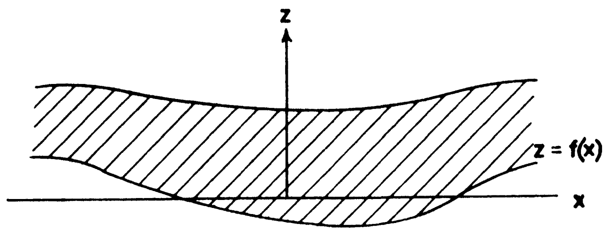


Fig. 24

Since the velocity $q$ is uniform and horizontal over any cross-section, the equation of continuity gives

$$qd = \text{constant} = h\sqrt{\alpha^3 gh} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The potential $V$ is $gz$, and, since the pressure is constant on the free surface, Bernouilli's equation (3.1.1) gives

$$\tfrac{1}{2}q^2 + V = \text{constant on the free surface}$$

i.e.
$$\tfrac{1}{2}q^2 + g(d + f(x)) = \text{constant} = \tfrac{1}{2}\alpha^3 gh + gh \quad . \quad . \quad (2)$$

Substituting for $q$ from (1) in (2) gives

$$\frac{\alpha^3 h^3}{2d^2} + d + f(x) = (\tfrac{1}{2}\alpha^3 + 1)h$$

i.e.
$$f(x) = (\tfrac{1}{2}\alpha^3 + 1)h - d - \frac{\alpha^3 h^3}{2d^2}$$

The maximum possible value of $f(x)$ is attained when the derivative with respect to $d$ vanishes, i.e. when

$$-1 + \alpha^3 h^3/d^3 = 0, \text{ i.e. } d = \alpha h \quad . \quad . \quad . \quad (3)$$

This is the only turning-point, and is a maximum (since the second derivative is negative). Substituting,

$$\begin{aligned}\text{maximum } f(x) &= (\tfrac{1}{2}\alpha^3 + 1)h - \tfrac{3}{2}\alpha h \\ &= \tfrac{1}{2}h(\alpha - 1)^2(\alpha + 2)\end{aligned}$$

Far downstream, i.e. as $x \longrightarrow \infty$, $f(x) \longrightarrow 0$, and the depth $d$ is $nh$, and so

$$\frac{\alpha^3}{2n^2} + n = \tfrac{1}{2}\alpha^3 + 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This is a cubic in $n$, of which one root is $n = 1$ (since upstream $d = h$). The other roots are then given by

$$2n^2 - \alpha^3 n - \alpha^3 = 0. \qquad (5)$$

The roots of this are of different signs, and hence, when $f(x) = 0$, a possible depth is $nh$, where $n$ is the positive root of this equation.

Now if the bed of the channel has a maximum level at one point only, i.e. if $f(x)$ has only one maximum (and no minimum, since the equation (3) shows there is only one turning-point), then $f(x) = 0$ at two points only, which are given to be $\pm\infty$.

Upstream the depth is given as $h$, and downstream, by (4), it is $nh$ where $n$ is either 1 or the positive root of equation (5). Consider the position if $n = 1$. In this case the stream would have the same velocity, namely $\sqrt{\alpha^3 gh}$ for $x = \pm\infty$. But, since the velocity is not everywhere constant, this means that there is at least one point where the velocity is either a maximum or minimum. However, it is known (see, e.g., Milne-Thompson, *Theoretical Hydrodynamics*, p. 86) that in irrotational motion under conservative forces the velocity cannot have either a maximum or a non-zero minimum within the fluid. Hence it is impossible to have the downstream depth the same as the upstream, and the downstream depth must be $nh$, where $n$ is the positive root of (5).

---

**Comment.** This is a straightforward application of the principles of continuity and energy (Bernouilli's equation) to a simple system. The final argument, however, requires knowledge of the impossibility of a non-zero stationary value of speed within a liquid.

## PROBLEMS FOR SOLUTION

1. In an infinite uniform liquid of density $\rho$ there is a spherical cavity of radius $a$ containing gas at rest at a pressure $\pi$. At time $t = 0$ the gas in the cavity immediately begins to expand adiabatically according to the law $pV^{4/3} =$ constant, where $V$ is the volume of the cavity and $p$ the gas pressure. Given that the pressure of the liquid at infinity vanishes and that gravity and the inertia of the gas are neglected, prove that the radius $R$ of the cavity at time $t$ is given by $R = a(1 + x)$, where

$$(1 + \tfrac{2}{3}x + \tfrac{1}{5}x^2)\sqrt{2x} = \frac{t}{a}\left(\frac{\pi}{\rho}\right)^{1/2} \qquad \text{(E.)}$$

2. An infinite mass of incompressible liquid is instantaneously at rest subject to a uniform pressure $p_0$, and it contains a spherical cavity of radius $a$ filled with gas at pressure $mp$. The inertia of the gas can be neglected, and throughout the subsequent motion the gas pressure is proportional to $\rho^\gamma$, where $\rho$ is density and $\gamma$ is a constant. Show that the radius of the sphere will oscillate between $a$ and $na$, where

$$\frac{m}{\gamma - 1}\left(1 - \frac{1}{n^{3\gamma-3}}\right) = n^3 - 1 \qquad \text{(D.)}$$

3. In an irrotational motion of inviscid liquid, constant density $\rho$ there is a source with rate of flow $4\pi Ua^2$ at the origin $O$ together with a uniform stream with velocity

$U$; the direction of $U$ is such that the point whose spherical coordinates are $(a, 0, 0)$ is a stagnation point. Show that the velocity at any point of the surface $r = a \sec \frac{1}{2}\theta$ is tangential to the surface, and if there is no body force, find the pressure at any point on this surface. (L.)

4. A sphere of radius $R$, whose centre is at rest, vibrates radially in an infinite incompressible fluid of density $\rho$ which is at rest at infinity. If the pressure at infinity is $\Pi$, show that the pressure $p$ at the surface of the sphere at time $t$ is

$$p = \Pi + \tfrac{1}{2}\rho\left\{\frac{d^2}{dt^2}(R^2) + \left(\frac{dR}{dt}\right)^2\right\}$$

If $R = a(2 + \cos nt)$, show that, to prevent cavitation in the fluid, $\Pi$ must not be less than $3\,\rho a^2 n^2$. (H.)

5. A uniform liquid rotates with constant angular velocity $\omega$ about a vertical axis. Show that the surfaces of equi-pressure are given by $r^2 - 2gz/\omega^2 =$ constant, where $z$ is drawn upwards and $r^2 = x^2 + y^2$.

A solid sphere of radius $a$ and weight $W$ floats in such a rotating liquid, and the free surface meets the sphere in the horizontal great circle (i.e. in the diametral plane). Find the density of the liquid. (D.)

6. A fluid is in equilibrium under the action of a field of force $\mathbf{F}$ per unit mass. Obtain the relation

$$\operatorname{grad} p = \rho\mathbf{F}$$

where $p$ denotes the pressure and $\rho$ the density.

A cylindrical vessel, of radius $a$ and height $h$, is closed at one end and open at the other end. The vessel stands on its base, with its axis vertical, and is just filled with liquid. If the liquid is made to rotate about the axis of the vessel with uniform angular velocity $\omega$, show that the volume of liquid spilt over the edge is $\pi\omega^2 a^4/4g$ provided that $a^2\omega^2 \leqslant 2gh$. Explain briefly what happens if $a^2\omega^2 > 2gh$. (H.)

CHAPTER 4

# ELECTRIC CURRENTS IN NETWORKS

The problems solved in this chapter require a knowledge of Ohm's law, Kirchhoff's laws, and the simplest facts about electromagnetic induction.

The differential equations arising are not solved in detail, and the student should verify solutions by whatever method he normally uses.

Problems on alternating currents are solved by using complex quantities, but only the simplest properties of complex numbers are required.

## 4.1 Basic Results

**4.1.1 Ohm's Law.** If $V$ is the potential difference between two points of a wire and $i$ is the current in the wire, then $V = Ri$, where $R$ is the resistance between the two points.

**4.1.2 Kirchhoff's Laws.** (i) The algebraic sum of the currents entering a junction is zero.

(ii) In any closed circuit the algebraic sum of the products of the resistance and current in each conductor is equal to the algebraic sum of the electromotive forces in the circuit.

**4.1.3 Rate of Generation of Heat (Dissipation of Energy) in the Circuit.** If $V$ is the potential difference, $R$ the resistance, and $i$ the current between two points of a conductor the rate of generation of heat is

$$Vi = Ri^2$$

## 4.2 Electromagnetic Induction

**4.2.1** If a loop of a circuit has self-inductance $L$, then, when a current $i$ flows in the loop, a "back" e.m.f. $-L\dfrac{di}{dt}$ is induced in the loop, the sense being such as to oppose the tendency of the current to increase.

**4.2.2** If the mutual inductance of two circuits $A$, $B$ is $M$ and currents $i_1$, $i_2$ flow in $A$, $B$ respectively, the e.m.f. induced in $A$ is $-M\dfrac{di_2}{dt}$ and in $B$ is $-M\dfrac{di_1}{dt}$.

**4.2.3** If $L_1$, $L_2$ are the self-inductances of two loops and $M$ their mutual inductance, then $M \leqslant \sqrt{L_1L_2}$. In practice, $M < \sqrt{L_1L_2}$.

## 4.3 Alternating Currents

If an alternating e.m.f. $E_0 \cos pt$ is applied to a circuit the complex solution of the resulting differential equation consists of the complementary function (the transient), which rapidly becomes negligible with time, and the particular integral (steady-state solution), this latter being the subject of the present section.

Since we are concerned only with the forced oscillations, all oscillating quantities may be assumed to have the same frequency $p/2\pi$, but there may be phase differences. In these problems the differential equations may be avoided, and the working shortened, by using complex quantities.

**4.3.1** When an alternating e.m.f. $E_0 \cos pt$ is applied to a circuit we may treat it as the real part of a complex e.m.f. $E = E_0 e^{ipt}$ (here $i$ is $\sqrt{-1}$). If the circuit includes resistance $R$, self-inductance $L$, and capacitance $C$ it may be shown that the steady-state current is the real part of

$$I = \frac{E_0 e^{ipt}}{R + i\left(Lp - \dfrac{1}{pC}\right)} = \frac{E}{Z}, \text{ say.}$$

$I$ is called the complex current and $Z$ the complex impedance of the circuit.

**4.3.2** By using complex current, e.m.f. and impedance the theory of networks, based on Ohm's law and Kirchhoff's laws, can be applied to a–c networks in the steady-state. Complex impedances may be combined in series and parallel by the same laws as for (real) resistances. Thus the equivalent impedance $Z$ of a circuit containing impedances $Z_1, \ldots Z_n$ in series is given by

$$Z = \sum_k Z_k$$

and, similarly, for impedances in parallel,

$$\frac{1}{Z} = \sum_k \frac{1}{Z_k}$$

**4.3.3** Students often find it easier, especially with inductively coupled circuits, to use the following rules:

the (complex) potential drop in a circuit carrying (complex) current $I$ due to

(i) a resistance $R$, is $RI$;
(ii) a self-inductance $L$, is $ipLI$;
(iii) a capacitance $C$, is $I/ipC = -iIpC$;
(iv) a mutual inductance $M$ from a coupled circuit with (complex) current $I^1$, is $ipMI^1$.

Applying Kirchhoff's second law (using these rules) to a circuit results in each term of the differential equation being replaced by one of the above. (See Problems 51, 53, 55.)

In general, the current will be out of phase with the e.m.f., and we may write $I = |I|\, e^{i(pt+\delta)}$, where $\delta$ is the phase shift.

**4.3.4 Mean Rate of Dissipation of Energy.** The actual current in a circuit loop, being the real part of the complex current, will have the form $|I| \cos(pt + \delta)$. If the loop contains an e.m.f. $E_0 \cos pt$ the power being expended at any instant is given by (from 4.1.3)

$$\begin{aligned} P &= E_0\,|I| \cos pt \cos(pt+\delta) \\ &= \tfrac{1}{2}E_0\,|I|\,(\cos\delta + \cos 2pt\cos\delta - \sin 2pt \sin\delta) \end{aligned}$$

If we find the average power $\overline{P}$ over a whole cycle, by integrating $P$ w.r.t. $t$ between limits 0 and $2\pi/p$ and dividing by $2\pi/p$ we get

$$\overline{P} = \tfrac{1}{2}E_0\,|I| \cos\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the resistance (real) of the loop is $R$ the power at any instant is (from 4.1.3),

$$R \times (\text{real part of } I)^2 = R\,|I|^2 \cos^2(pt+\delta)$$

Averaging this in the same way over a whole cycle gives

$$\overline{P} = \frac{p}{2\pi} R\,|I|^2 \int_0^{2\pi/p} \cos^2(pt+\delta)\,dt = \tfrac{1}{2}R\,|I|^2 \quad . \quad . \quad (2)$$

---

**Problem 46**

Twelve straight uniform wires of equal length $a$ and equal resistance $r$ are joined to form the edges of a cube of side $a$. Current enters the system at one vertex. Show that

(i) if the current leaves at the opposite vertex the resistance of the system is $\dfrac{5r}{6}$;

(ii) if the current leaves from a point on one of the edges meeting at the opposite vertex and at a distance $xa$ $(0 \leqslant x \leqslant 1)$ from it, the resistance of the system is a quadratic function of $x$. If $x$ is chosen so as to make the resistance a maximum, show that

$$x = \frac{2}{5} \text{ and the resistance is } \frac{9r}{10} \qquad \text{(L.)}$$

---

**Solution.**

(i) Considerations of symmetry and Kirchhoff's first law (4.1.2) give the current distribution shown. Then the potential drop from $A$ to $G$ (along the path $ABFG$, say) is

$$r\left(\frac{i}{3}+\frac{i}{6}+\frac{i}{3}\right)=\frac{5r}{6}i$$

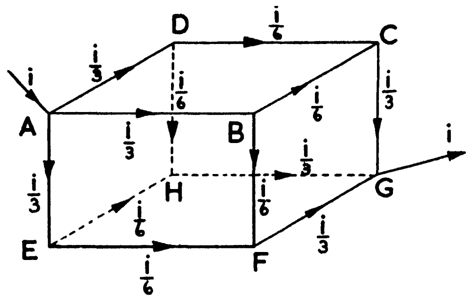


Fig. 25

and the equivalent resistance (4.1.1) is

$$\frac{5ri}{6}/i=\frac{5r}{6}$$

(ii) By symmetry and the application of Kirchhoff's first law (4.1.2) the current distribution can be expressed in terms of three unknown currents $w$, $y$, $z$ as shown, the total current through the network being $i = 2y + z$.

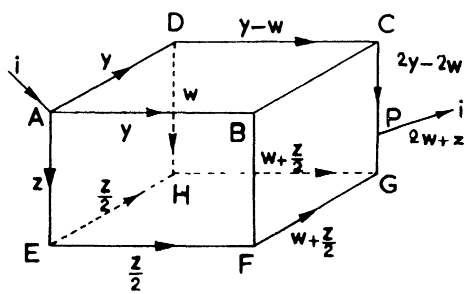


Fig. 26

Round the circuit $ABCP$ we find, for the potential drop from $A$ to $P$ (4.1.1),

$$V_A - V_P = r\left\{3y-\frac{3z}{2}+(1-x)(4y-3z)\right\} \quad . \quad . \quad (1)$$

Similarly, round $AEFG$

$$V_A - V_P = r\left\{\frac{7z}{2}-y+x(4z-2y)\right\} \quad . \quad . \quad . \quad (2)$$

Finally, since there is no potential drop round $ADHEA$, we find, after dividing by $r$,

$$\frac{3z}{2} = w + y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Using (3) to eliminate $w$ and then equating (1) and (2) gives

$$z = \frac{2y(4 - x)}{8 + x}$$

Hence the total current through the network is

$$i = 2y + z = 2y\left(\frac{12}{8 + x}\right)$$

Also substituting for $w$ and $z$ in terms of $y$,

$$V_A - V_P = \frac{ry}{8 + x}(20 + 8x - 10x^2)$$

Hence the equivalent resistance of the network is

$$R = \frac{V_A - V_P}{i} = \frac{r}{12}(10 + 4x - 5x^2)$$

Simple calculus shows that this is a maximum when $x = \frac{2}{5}$ and, for this value of $x$, $R = 9r/10$.

---

**Problem 47**

A network of conducting wires forms the edges of a tetrahedron $ABCD$. The resistance of the wire $AB$ is $x$ and can be varied. The resistance of each of the other five wires is constant and equal to $r$. The two vertices $A$ and $C$ are maintained at a constant potential difference $V$. Show that the heat loss in $AB$ is a maximum when $x = \dfrac{3r}{5}$, and show that the heat loss in the network as a whole decreases steadily as $x$ increases and lies between $8V^2/5r$ and $8V^2/3r$. (L.)

---

**Solution.**

Let the currents flowing along $AB$, $AC$, $AD$ be $a$, $b$, $c$ and along $DC$ be $y$.

Then by applying Kirchhoff's first law (4.1.2) to the points $B$ and $D$, we find the currents in $DB$ and $BC$ to be $c - y$ and $a + c - y$.

Then the second law (4.1.2) applied to the circuit $BCD$ gives

$$a + 2c - 3y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The potential drop from $A$ to $C$ is $V$, and if we build this up as the sum of the products of resistance and current (4.1.1) along the three paths $AC$, $AB + BC$, $AD + DC$ we get

$$V = br \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$= ax + (a + c - y)r \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$= cr + yr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Eliminating $c$ from (1) and (4) gives $y = \frac{1}{5}\left(a + \frac{2V}{r}\right)$

$$\therefore \qquad c = \frac{V}{r} - y = \frac{3V}{5r} - \frac{1}{5}a$$

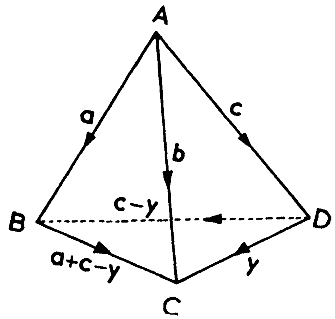


Fig. 27

Substituting for $y$ and $c$ in (3) gives

$$a = \frac{4V}{5x + 3r}$$

Thus the rate of heat production in $AB$ (4.1.3) is

$$\frac{16V^2x}{(5x + 3r)^2}$$

and straightforward differentiation shows this to be a maximum when $x = 3r/5$.

---

To calculate the heat loss in the whole network it is quickest to use the product of the potential drop $V$ from $A$ to $C$ and the current, $(a + b + c)$, flowing through the network.

Thus heat loss $= V(a + b + c)$, which, after substitution and reduction, gives

$$\frac{8V^2(r + x)}{r(5x + 3r)}$$

The derivative of this is $-\frac{16v^2}{(3r+5x)^2} < 0$ so that the heat loss is a continually decreasing function of $x$ and the extreme values, when $x = 0$ and as $x \longrightarrow \infty$, are $\frac{8V^2}{3r}$ and $\frac{8V^2}{5r}$.

---

**Problem 48**

A telegraph wire $A_0A_1A_2 \ldots A_n$ has a number of faulty connections at $A_1, A_2, \ldots A_{n-1}$ through which the current leaks to earth. The portion of the wire between $A_k$ and $A_{k+1}$ $(0 \leqslant k \leqslant n-1)$ has a resistance $r$, while the resistance between $A_k$ $(1 \leqslant k \leqslant n-1)$ and the earth is $R$. Prove that the potential $V_k$ of the fault $A_k$ is

$$V_k = \frac{V_0 \sinh (n-k)\theta + V_n \sinh k\theta}{\sinh n\theta}$$

where $\cosh \theta = 1 + \frac{r}{2R}$.

If the terminal $A_n$ is earthed, and $n\theta$ is large, show that the effective resistance between $A_0$ and the earth is approximately

$$\tfrac{1}{2}[r + (r^2 + 4rR)^{1/2}] \qquad \text{(L.)}$$

---

**Solution.**

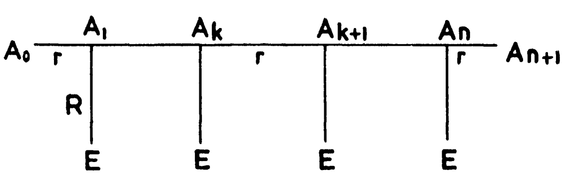


Fig. 28

Let the current flowing from $A_k$ to $A_{k+1}$ be $i_k$.
Then, by Ohm's law (4.1.1),

$$i_k = \frac{V_k - V_{k+1}}{r} \quad (0 \leqslant k \leqslant n-1) \quad . \quad . \quad . \quad (1)$$

Also, equating the two expressions, from Kirchhoff's first law (4.1.2) and from Ohm's law (4.1.1), for the current flowing to earth through the fault at $A_k$, we have

$$i_{k-1} - i_k = \frac{V_k}{R} \qquad (1 \leqslant k \leqslant n-1) \quad . \quad . \quad . \quad (2)$$

Using (1) to replace $i_{k-1}$ and $i_k$ in (2) gives

$$\frac{V_{k-1} - V_k}{r} - \frac{V_k - V_{k+1}}{r} = \frac{V_k}{R}$$

i.e. $$V_{k+1} - V_k\left(2 + \frac{r}{R}\right) + V_{k-1} = 0 \quad (1 \leqslant k \leqslant n-1)$$

or $$V_{k+1} - 2V_k \cosh \theta + V_{k-1} = 0$$

The solution of this difference equation is

$$V_k = Ap^k + Bq^k$$

when $A$ and $B$ are constants and $p$ and $q$ are the roots of

$$x^2 - 2x \cosh \theta + 1 = 0$$

i.e. $$p = e^\theta, q = e^{-\theta}$$

Hence $$V_k = A\, e^{k\theta} + Be^{-k\theta}$$
$$= L \cosh k\theta + M \sinh k\theta$$

---

Putting $k = 0$ and $k = n$ to find the arbitrary constants $L$, $M$ gives, as required,

$$V_k = \frac{V_0 \sinh (n-k)\,\theta + V_n \sinh k\theta}{\sinh n\theta} \quad . \quad . \quad . \quad (3)$$

---

When $A_n$ is earthed $V_n = 0$ and all the current $i_1$ entering at $A_0$ goes to earth. This is given by (from (1))

$$i_1 = i = \frac{V_0 - V_1}{r} = \frac{1}{r}\left[V_0 - \frac{V_0 \sinh (n-1)\,\theta}{\sinh n\theta}\right]$$

substituting for $V_1$ from (3). Then the effective resistance between $A_0$ and earth

$$= \frac{V_0}{i} = \frac{r \sinh n\theta}{\sinh n\theta - \sinh (n-1)\theta}$$

When $n\theta$ is large, we may neglect $e^{-n\theta}$ and $e^{-(n-1)\theta}$, and the resistance becomes approximately

$$\frac{re^{n\theta}}{e^{n\theta} - e^{(n-1)\theta}} = \frac{re^\theta}{e^\theta - 1}$$

Since $\cosh \theta = 1 + \frac{r}{2R}$, we find $e^\theta = 1 + \frac{1}{2R}(r + \sqrt{r^2 + 4Rr})$.

The expression for the resistance then reduces to $r + \frac{2Rr}{r + \sqrt{r^2 + 4Rr}}$, and rationalisation of the fraction gives the stated result.

---

**Note.** When $n$ is large a rather interesting alternative derivation of the last part is as follows.

The resistances $r$ from $A_{n-1}$ to $A_n$ and $R$ from $A_{n-1}$ to earth are in parallel, and thus have a combined resistance, $r_1$, say, where

$$r_1 = \left(\frac{1}{r} + \frac{1}{R}\right)^{-1}$$

Now $r_1$ and $r$ (from $A_{n-2}$ to $A_{n-1}$) are in series and give

$$r_2 = r + \left(\frac{1}{r} + \frac{1}{R}\right)^{-1},$$

$r_2$ is in parallel with $R$, and so on. In fact, proceeding in this way, we can express the net resistance of the circuit as the continued fraction

$$r + \frac{1}{R^{-1}+}\,\frac{1}{r+}\,\frac{1}{R^{-1}+}\,\frac{1}{r+}\cdots\frac{1}{r+}\,\frac{1}{R^{-1}+}\,\frac{1}{r}$$

If $n$ is large we can treat this as an infinite continued fraction whose value, $x$, satisfies

$$x = r + \frac{1}{\dfrac{1}{R} + \dfrac{1}{x}} = r + \frac{x}{\dfrac{x}{R} + 1}$$

This gives a quadratic equation whose positive root is

$$\tfrac{1}{2}[r + (r^2 + 4Rr)^{1/2}]$$

---

**Problem 49**

A circuit is made up of two concentric regular polygons each of $n$ sides, with vertices $A_1, A_2, \ldots A_n$ and $B_1, B_2, \ldots B_n$ respectively (and so placed that the lines $A_1B_1, A_2B_2, \ldots A_nB_n$ meet in the centre $O$) together with cross pieces $A_1B_1, \ldots A_nB_n$. The material of the circuit is such that the resistance of each of the wires $A_1A_2, \ldots A_nA_1$, $B_1B_2, \ldots B_nB_1$, $A_1A_1, \ldots A_nB_n$ is the same and equal to $r$. Current enters the circuit at $A_1$ and leaves at $B_1$. Show that the current in $A_rA_{r+1}$ is $C \sinh(\alpha r - \frac{1}{2}n\alpha)$, where $C$ is a constant, and that the equivalent resistance of the circuit is

$$\frac{-\sqrt{3}\cosh\frac{1}{2}n\alpha + \sinh\frac{1}{2}n\alpha}{\sqrt{3}\cosh\frac{1}{2}n\alpha - 3\sinh\frac{1}{2}n\alpha}\, r$$

where $\cosh\alpha = 2$. (D.)

---

**Solution.**

Let $I$ be the total current entering at $A_1$ and leaving at $B_1$ and let $i_r$ be the current circulating in the mesh $A_rA_{r+1}B_{r+1}B_r$, so that Kirchhoff's first law (4.1.2) is automatically satisfied.

Then applying the second law (4.1.2) to this $r$th mesh leads at once to the difference equation

$$i_{r+1} - 4i_r + i_{r-1} = 0$$

Putting $\cosh \alpha = 2$ gives (as in Problem 48) the solution

$$i_r = Ae^{r\alpha} + Be^{-r\alpha} \qquad (r = 1, \ldots n) \quad . \quad . \quad . \quad (1)$$

where $A$ and $B$ are constants.

From the given conditions the current distribution is clearly symmetrical about $A_1B_1$, and hence either

$$i_{\frac{1}{2}n} = -i_{\frac{1}{2}n} = 0 \quad (n \text{ even})$$

or

$$i_{\frac{1}{2}(n-1)} = -i_{\frac{1}{2}(n+1)} \quad (n \text{ odd})$$

and substitution in (1) gives $B = -Ae^{n\alpha}$. Thus (1) becomes

$$i_r = Ae^{r\alpha} - Ae^{-r\alpha+n\alpha} = C \sinh (r\alpha - \tfrac{1}{2}n\alpha)$$

where

$$C = 2Ae^{\frac{1}{2}n\alpha} \text{ (constant)}$$

Thus

$$i_r = C \sinh (r\alpha - \tfrac{1}{2}n\alpha) \quad . \quad . \quad . \quad . \quad (2)$$

For the first mesh we find, applying Kirchhoff's second law and remembering that $i_n = -i_1$, by symmetry,

$$5i_1 - i_2 = I \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The current in $A_1B_1$ is $I + i_n = I - 2i_1$, so that the potential drop from $A_1$ to $B_1$ is $r(I - 2i_1)$, and hence the equivalent resistance of the circuit (4.1.1) is $(I - 2i_1)r/I$

$$= \frac{3i_1 - i_2}{5i_1 - i_2} r, \text{ by (3)}$$

$$= \frac{3 \sinh (1 - \frac{1}{2}n)\alpha - \sinh (2 - \frac{1}{2}n)\alpha}{5 \sinh (1 - \frac{1}{2}n)\alpha - \sinh (2 - \frac{1}{2}n)\alpha} r \text{ by (2)}$$

Expanding this in terms of $\sinh \frac{1}{2}n\alpha$ and $\cosh \frac{1}{2}n\alpha$ and using $\cosh \alpha = 2$ leads at once to the required result.

---

### Problem 50

A circuit contains a condenser of capacity $C$, a wire of resistance $R$, and a coil of self-inductance $L = 2R^2C$ and negligible resistance. The coil is in parallel with the wire and the condenser is in series with them. At time $t = 0$ the charge of the condenser is $Q_0$ and there is no current passing through the coil. Show that the charge of the condenser at a later time is

$$Q_0\, e^{-kt}(\cos kt - \sin kt), \text{ where } k = 1/2RC \qquad \text{(C.)}$$

---

**Solution.**

Let $Q$ be the charge of the condenser at time $t$ and the currents be as shown. Then

$$i_1 + i_2 = \frac{dQ}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The potential drop across the condenser is $\frac{Q}{C}$ and across $L$ is $L\frac{di_2}{dt}$.

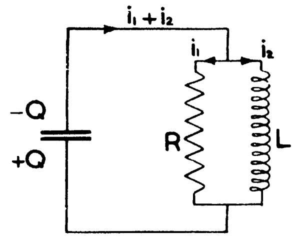


Fig. 29

Hence, by Kirchhoff's second law,

$$\frac{Q}{C} + L\frac{di_2}{dt} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Similarly, $$Ri_1 - L\frac{di_2}{dt} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Using (3) and the result of differentiating (2) to eliminate $i_1$ and $Q$ from (1) gives

$$\frac{d^2i_2}{dt^2} + \frac{1}{CR}\frac{di_2}{dt} + \frac{1}{CL}i_2 = 0$$

The solution of this using $L = 2R^2C$, is

$$i_2 = e^{-kt}(A \cos kt + B \sin kt)$$

where $k = 1/2RC$. When $t = 0, i_2 = 0$, whence $0 = A$.

$\therefore$ $$i_2 = Be^{-kt} \sin kt$$

and, from (2),

$$Q = -CL\frac{di_2}{dt} = -CL\, e^{-kt}B\,(-k \sin kt + k \cos kt)$$

When $t = 0$, $Q = Q_0$. Hence $Q_0 = -BCL\,k$ and $Q = Q_0\, e^{-kt}$ (cos $kt$ — sin $kt$) as required.

---

**Problem 51**

Define the self-inductances $L_1$, $L_2$ and the mutual inductance $M$ of two closed circuits $C_1$, $C_2$ in free space and state the inequality which they satisfy.

A circuit $C_1$ consists of an inductance $L_1$ in series with a resistance $R_1$, and a circuit $C_2$ consists of an inductance $L_2$ in series with a resistance $R_2$. The mutual inductance of $C_1$ and $C_2$ is $M$. If the circuit $C_1$ contains an e.m.f. $E_0 \exp(i\omega t)$, find the value of $\omega$ for which the current amplitude in $C_2$ is a maximum. Find also this maximum, and show that it cannot be greater than $E_0/2(R_1R_2)^{1/2}$ for any admissible values of $L_1$, $L_2$, and $M$. (L.)

**Solution.**

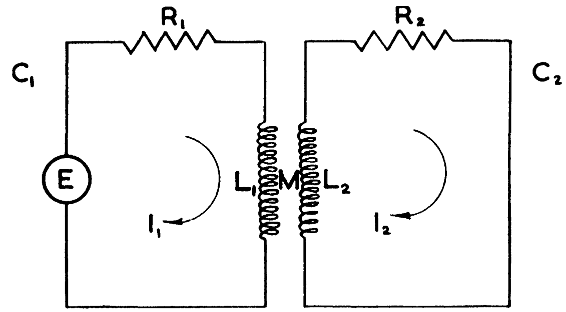


Fig. 30

We avoid solving the differential equations of the circuits by using complex impedances. This amounts to combining in series the (complex) resistances

$$R_1,\ i\omega L_1,\ i\omega M\ I_2/I_1$$

and

$$R_2,\ i\omega L_2,\ i\omega M\ I_1/I_2$$

in $C_1$ and $C_2$ respectively, where $I_1$ and $I_2$ are the (complex) currents in each circuit.

Then applying Kirchhoff's second law to each circuit gives, for $C_1$,

$$(R_1 + i\omega L_1)\,I_1 + i\omega M I_2 = E \quad . \quad . \quad . \quad . \quad (1)$$

where $E = E_0 \exp(i\omega t)$, and, for $C_2$,

$$i\omega M I_1 + (R_2 + i\omega L_2)\,I_2 = 0 \quad . \quad . \quad . \quad . \quad (2)$$

Solving (1) and (2) for $I_2$ gives

$$I_2 = \frac{i\omega M E_0\, e^{i\omega t}}{A - iB} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where

$$A = \omega^2 (L_1L_2 - M^2) - R_1R_2 \quad . \quad . \quad . \quad . \quad (4)$$
$$B = \omega\,(R_1L_2 + R_2L_1)$$

Rationalising (3) and picking out its real part gives for the (real) current in $C_2$

$$-\frac{\omega M E_0\,(A \sin \omega t + B \cos \omega t)}{A^2 + B^2}$$

This is a variable current whose amplitude is clearly

$$\frac{\omega M E_0}{\sqrt{A^2 + B^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Differentiating this w . r . t . $\omega$, using (4) and equating the differential coefficient to zero, gives

$$A^2 - 2\omega^2 A(L_1L_2 - M^2) = 0$$

Thus, either $A = 0$, whence $\omega^2 = R_1R_2/(L_1L_2 - M^2)$

or $A - 2\omega^2(L_1L_2 - M^2) = 0$

i.e. $\omega^2(L_1L_2 - M^2) + R_1R_2 = 0$, which is not possible, as $L_1L_2 \geqslant M^2$.

Substituting the result $A = 0$, which gives the maximum amplitude, in (5) we find the value of this maximum to be

$$\frac{\omega M E_0}{B} = \frac{M E_0}{R_1L_2 + R_2L_1} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Now put $M = k\sqrt{L_1L_2}$ $(k \leqslant 1)$. Substituting in (6) gives

$$\frac{kE_0 \sqrt{L_1L_2}}{R_1L_2 + R_2L_1}$$

The reciprocal of this may be written in the form

$$\frac{1}{kE_0}\left(R_1\sqrt{\frac{L_2}{L_1}} + R_2\sqrt{\frac{L_1}{L_2}}\right)$$

and taking $\frac{L_2}{L_1}$ to be the variable, this is easily shown to be a *minimum* when $\frac{L_2}{L_1} = \frac{R_2}{R_1}$.

This gives $\frac{kE_0}{2\sqrt{(R_1R_2)}}$ as the maximum value of (6), which, as $k \leqslant 1$, is the result required.

---

**Problem 52**

A coil of inductance $L$, a condenser of capacity $C$, and a battery of electromotive force $E$ and resistance $R$ are connected in parallel. Assuming that all resistances other than that of the battery can be neglected and that $L < 4CR^2$, show that the current through the coil at time $t$ after the battery connection is made is given by

$$i_1 = \frac{E}{R}\left\{1 - e^{-t/2RC}\left(\cos \omega t + \frac{1}{2\omega RC}\sin \omega t\right)\right\}$$

where
$$\omega^2 = \frac{1}{LC} - \frac{1}{4R^2C^2}$$

Also show that the charge on the condenser at this time is

$$Q = \frac{E}{R\omega}\, e^{-t/2RC} \sin \omega t \qquad \text{(E.)}$$

---

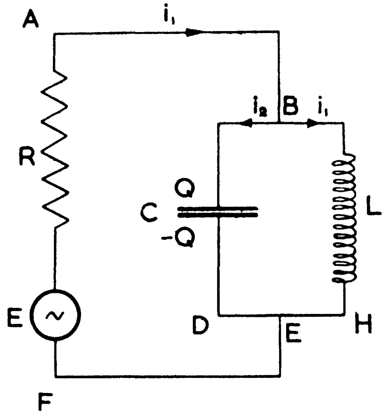


Fig. 31

Let the currents be $i$, $i_1$, $i_2$, as shown, and let $Q$ be the charge on the condenser, at time $t$.

By Kirchhoff's first law,

$$i = i_1 + i_2 \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

By the second law applied to the circuit $ABGHFA$,

$$E - L\frac{di_1}{dt} = Ri = R(i_1 + i_2) \qquad . \quad . \quad . \quad . \quad (2)$$

Similarly, for the circuit $ABCDEFA$,

$$E - \frac{Q}{c} = Ri = R(i_1 + i_2) \qquad . \quad . \quad . \quad . \quad (3)$$

Also $\qquad dQ/dt = i_2 \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$

Comparison of (2) and (3) gives $Q = CL\dfrac{di_1}{dt}$, which, with (4), gives $i_2 = CL\dfrac{d^2i_1}{dt^2}$. Substituting in (2) and rearranging gives

$$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right)i_1 = \frac{E}{LRC}$$

where $D = d/dt$.

A particular integral is clearly $E/R$ and the complementary function is $e^{-t/2RC}(A \cos \omega t + B \sin \omega t)$, where $A$, $B$ are constants and

$$\omega^2 = \frac{1}{LC} - \frac{1}{4R^2C^2}$$

$$\therefore \qquad i_1 = E/R + e^{-t/2RC}(A \cos \omega t + B \sin \omega t)$$

When $t = 0$, $i_1 = 0$ and $Q = 0$; hence $\dfrac{di_1}{dt} = 0$, and

$$A = -\frac{E}{R}, \ B = -\frac{E}{2R^2C\omega}$$

and hence $$\underline{i_1 = \frac{E}{R}\left\{1 - e^{-t/2RC}\left(\cos \omega t + \frac{1}{2\omega RC} \sin \omega t\right)\right\}}$$

Also $Q = CLdi_1/dt$ leads at once to $Q = \dfrac{E}{R\omega} e^{-t/2RC} \sin \omega t$.

---

**Problem 53**

Describe briefly the use of impedance in investigating alternating current networks.

Points $A$ and $B$ are connected by a condenser of capacitance $C$; $B$ and $D$ by a wire of resistance $R$; $A$ and $F$ by a coil of inductance $L$; $B$ and $F$ by a wire of resistance $R$; and $D$ and $F$ by a wire of resistance $R$. An alternating electromotive force $E_0 \cos \omega t$ is applied between $A$ and $D$. Show that the amplitude of the current in $BF$ is

$$E_0(1 + \omega^2LC)/\{4R^2(1 - \omega^2LC)^2 + \omega^2(3L + CR^2)^2\}^{1/2}$$

and that the difference in phase between this current and the applied electromotive force is

$$\tan^{-1}\left\{\frac{\omega(3L + CR^2)}{2R(\omega^2LC - 1)}\right\} \qquad \text{(D.)}$$

---

**Solution.**

Using complex e.m.f., currents etc. (4.3) we may write down, by Kirchhoff's second law applied to the three meshes of the circuit as shown, the equations

$$\left.\begin{aligned} &i\omega L(I_3 - I_1) + R(I_3 - I_1) = E_0e^{i\omega t} \\ &(i\omega L - i/\omega C)I_1 - i\omega LI_3 + R(I_1 - I_2) = 0 \\ &R(3I_2 - I_1 - I_3) \qquad\qquad\qquad\quad = 0 \end{aligned}\right\}$$

Since we require only the current in $BF$, we eliminate $I_3$ and solve for $I_1$ and $I_2$, getting

$$I_1 = \frac{E_0(R + 3i\omega L)e^{i\omega t}}{R^2 + 3L/C + 2i\omega RL - i2R/\omega C}$$

$$I_2 = \frac{E_0\left(R + 2i\omega L - \dfrac{i}{\omega C}\right)e^{i\omega t}}{R^2 + 3L/C + 2i\omega RL - i2R/\omega C}.$$

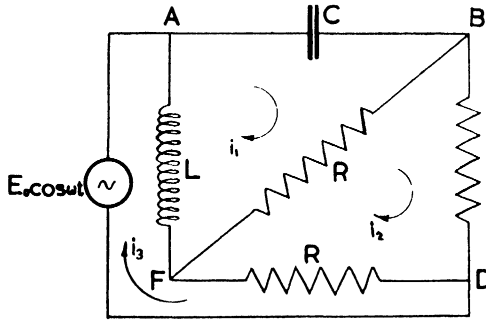


Fig. 32

Hence (after simplification)

$$I_1 - I_2 = \frac{i(1 + \omega^2 LC)E_0 e^{i\omega t}}{(3L + CR^2)\omega + 2iR(\omega^2 LC - 1)}$$

If we write

$$I_1 - I_2 = Ie^{i\omega t}$$

$I$ will be a complex number of the form $|I|e^{i\delta t}$, say, where $|I|$ is the amplitude of the current in $BF$ and $\delta$ the phase difference between this current and the applied e.m.f.

Thus
$$|I|e^{i\delta t} = \frac{i(1 + \omega^2 LC)E_0}{(3L + CR^2)\omega + 2iR(\omega^2 LC - 1)}$$

Hence
$$|I| = \frac{E_0(1 + \omega^2 LC)}{\{4R^2(1 - \omega^2 LC)^2 + \omega^2(3L + CR^2)^2\}^{1/2}}$$

and
$$\delta = \tan^{-1}\left\{\frac{\omega(3L + CR^2)}{2R(\omega^2 LC - 1)}\right\}$$

---

**Problem 54**

A circuit $A$ containing a switch and a battery providing constant electromotive force $E$ has resistance $R$ and self-induction $L$. A second closed circuit $B$ has resistance $R$ and self-induction $L$. The coefficient of

mutual induction of $A$ and $B$ is $L \sin \alpha$, ($\alpha \neq \frac{1}{2}\pi$). The switch in circuit $A$ is closed at time $t = 0$.

Find the currents in the circuits at time $t$, and show that the current in circuit $B$ reaches a maximum after a time

$$\frac{L \cos^2 \alpha}{2R \sin \alpha} \log \{(1 + \sin \alpha)/(1 - \sin \alpha)\}$$

and then approaches zero asymptotically.

What is the physical significance of the condition $\alpha \neq \frac{1}{2}\pi$? If $\alpha = \frac{1}{2}\pi$ show that the currents in $A$ and $B$ at time $t$ are

$$\frac{E}{R}(1 - \tfrac{1}{2}e^{-Rt/2L}) \text{ and } -\frac{E}{2R}e^{-Rt/2L}$$

respectively. (H.)

---

**Solution.**

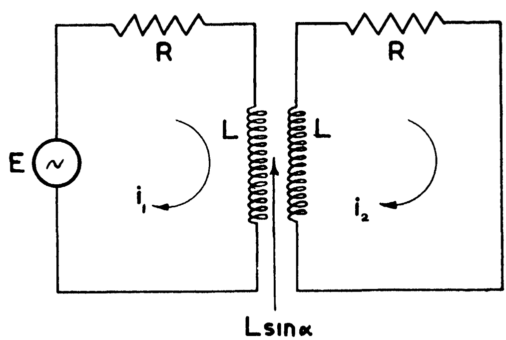


Fig. 33

Applying Kirchhoff's second law to each circuit gives

$$\left.\begin{aligned} Ri_1 + L\frac{di_1}{dt} + L \sin \alpha \frac{di_2}{dt} &= E \\ Ri_2 + L\frac{di_2}{dt} + L \sin \alpha \frac{di_1}{dt} &= 0 \end{aligned}\right\}$$

i.e.

$$(R + LD)i_1 + L \sin \alpha .\ Di_2 = E \quad . \quad . \quad . \quad (1)$$

$$L \sin \alpha .\ Di_1 + (R + LD)i_2 = 0 \quad . \quad . \quad . \quad (2)$$

whence

$$(L^2 \cos^2 \alpha .\ D^2 + 2RLD + R^2)i_1 = RE$$

from which we find

$$\underline{i_1 = \frac{E}{R} + Ae^{pt} + Be^{qt}} \quad . \quad . \quad . \quad . \quad (3)$$

where $$p = -\frac{R}{L(1 - \sin\alpha)},\ q = -\frac{R}{L(1 + \sin\alpha)} \qquad . \quad (4)$$

Eliminating $Di_2$ between (1) and (2) and using (3) leads to

$$i_2 = Be^{qt} - Ae^{pt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

At time $t = 0$, $i_1 = 0 = i_2$, and hence $A = B = -\dfrac{E}{2R}$

Thus

$$\left.\begin{aligned} i_1 &= \frac{E}{R}\{1 - \tfrac{1}{2}e^{pt} - \tfrac{1}{2}e^{qt}\} \\ i_2 &= \frac{E}{R}\{\tfrac{1}{2}e^{pt} - \tfrac{1}{2}e^{qt}\} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

When $i_2$ is a maximum, $di_2/dt = \dfrac{E}{2R}(pe^{pt} - qe^{qt}) = 0$

or $$e^{(q-p)t} = p/q.$$

Using (4), this leads at once to the result

$$t = \frac{L\cos^2\alpha}{2R\sin\alpha}\log\frac{1 + \sin\alpha}{1 - \sin\alpha}$$

---

$\alpha \neq \dfrac{\pi}{2}$ expresses the fact that in practice $M^2 < L_1L_2$, $M$ being the mutual inductance between two circuits of self-inductance $L_1$, $L_2$ respectively (4.2.3). If, in (3) and (5), $\alpha \longrightarrow \pi/2$, then, using (4), we find, in the limit,

$$i_1 = \frac{E}{R}\left(1 - \tfrac{1}{2}e^{-Rt/2L}\right)$$

and $$i_2 = -\frac{E}{2R}\,e^{-Rt/2L}$$

---

**Comment.** It should be noted that in this case the limit must be taken *after* the solution of the differential equation; the limiting currents do not in fact satisfy either of the original boundary conditions $i_1 = 0$, $i_2 = 0$, $t = 0$. If we put $\alpha = \dfrac{\pi}{2}$ before solving the equations (1), (2), then these equations simplify and the result of eliminating $i_2$ between them is to give a first-order equation in $i_1$ rather than the second-order equation which occurs in the general case; thus only one boundary condition can be satisfied, and if we use either of the original boundary conditions the final currents will differ from those obtained by the limiting process.

Inspection of the solution (3) shows that the limiting process $\alpha \longrightarrow \frac{\pi}{2}$ removes the term in $e^{pt}$ altogether, whereas, for $\alpha$ just less than $\frac{\pi}{2}$, $e^{pt}$ has the value 1 at $t = 0$ (so that the solution satisfies the boundary conditions at $t = 0$) but dies away rapidly with increasing $t$. The limiting solution obtained is thus the true limit for all $t \neq 0$ when $\alpha \longrightarrow \frac{\pi}{2}$ from below, and the justification for it is the physical fact that $\alpha < \frac{\pi}{2}$ for all real circuits. The question does not make it clear that the currents are limiting rather than actual ones.

---

**Problem 55**

Two circuits coupled by a mutual inductance $M$ are made up as follows. The first circuit contains, in series, resistance $R_1$, self-inductance $L_1$, and a generator producing a potential difference $V \cos \omega t$. The second circuit consists of resistance $R_2$, and self-inductance $L_2$. Show that $E_1$ and $E_2$, the mean rates of dissipation of energy in $R_1$ and $R_2$, respectively, satisfy

$$E_2/E_1 = \omega^2 M^2 R_2/(R_2^2 + \omega^2 L_2^2)R_1$$

and that $$E_2 = \tfrac{1}{2}V^2\omega^2M^2R_2\{R_2^2(R_1^2 + \omega^2L_1^2) + 2\omega^2M^2R_1R_2 + \omega^2[\omega^2(L_1L_2 - M^2)^2 + L_2^2R_1^2]\}^{-1}$$

Draw rough sketches indicating the behaviour of $E_2/E_1$ and $E_2$ as $R_2$ is varied and all other parameters are kept fixed. (L.)

---

**Solution.**

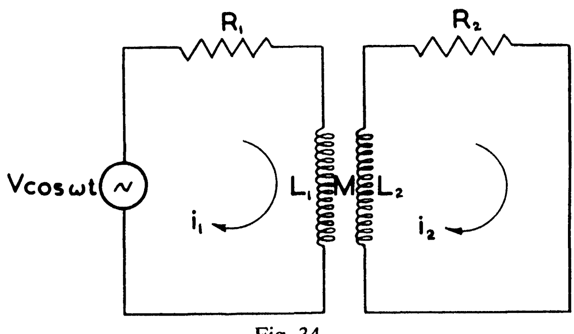


Fig. 34

Let $I_1$, $I_2$, be the complex currents in the circuits and $Ve^{i\omega t}$ the complex e.m.f. (4.3). Then Kirchhoff's second law gives

$$\left.\begin{aligned}(R_1 + i\omega L_1)I_1 + i\omega M I_2 &= Ve^{i\omega t}\\ (R_2 + i\omega L_2)I_2 + i\omega M I_1 &= 0\end{aligned}\right\}$$

Solving these gives

$$I_1 = \frac{(R_2 + i\omega L_2)Ve^{i\omega t}}{(R_1 + i\omega L_1)(R_2 + i\omega L_2) + \omega^2 M^2}$$

$$= \frac{(R_2 + i\omega L_2)Ve^{i\omega t}}{R_1R_2 + \omega^2(M^2 - L_1L_2) + i\omega(R_1L_2 + R_2L_1)}$$

and $$I_2 = \frac{-i\omega M\ Ve^{i\omega t}}{R_1R_2 + \omega^2(M^2 - L_1L_2) + i\omega(R_1L_2 + R_2L_1)}$$

The mean rate of dissipation of energy is, in $R_1$, $E_1 = \frac{1}{2}R_1|I_1|^2$ and, in $R_2$, $E_2 = \frac{1}{2}R_1|I_2|^2$ (4.3.4).
Hence, at once, from the previous results for $I_1$, $I_2$,

$$\frac{E_2}{E_1} = \frac{\omega^2 M^2 R_2}{(R_2{}^2 + \omega^2 L_2{}^2)R_1}$$

and $$E_2 = \frac{\frac{1}{2}\omega^2 M^2 V^2 R_2}{\{[R_1R_2 + \omega^2(M^2 - L_1L_2)]^2 + \omega^2(R_1L_2 + R_2L_1)^2\}}$$

which can be rearranged to give the stated answer.

---

If only $R_2$ may vary we can write

$$E_2/E_1 = AR_2/(R_2{}^2 + B),\ A = \omega^2 M^2/R_1,\ B = \omega^2 L_2.$$

Considering only positive values of $R_2$, we find $E_2/E_1$ has a maximum value of $\frac{1}{2}AB^{-1/2}$ at $R_2 = B^{1/2}$; its variation is indicated in the sketch.

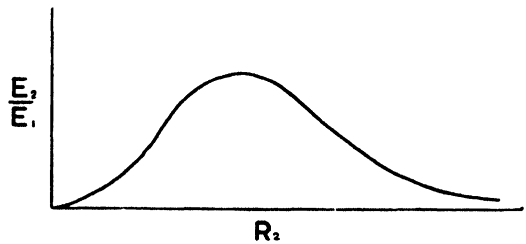


Fig. 35

Similarly, $E_2$ has the form

$$\frac{A^1R_2}{R_2{}^2 + B^1R_2 + C^1}$$

and varies as shown.

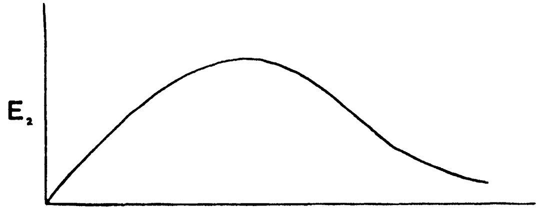


Fig. 36

## PROBLEMS FOR SOLUTION

1. State the laws governing the distribution of steady currents in a network of linear conductors.

Six equal wires, each of resistance $r$, form a regular tetrahedron. Current is led into the network at a vertex and is withdrawn from a point in one of the opposite sides.

Show that the effective resistance lies between $r/2$ and $5r/8$.

Find also the maximum rate of heat production. (L.)

2. A rectangle $ABCD$ has sides $AD$, $BC$ of length $3\lambda$ and sides $AB$, $DC$ of length unity. $E$, $F$ and $G$, $H$ are points of trisection of $AD$ and $BC$ respectively and $EG$, $FH$ are lines parallel to $AB$. The lines in the figure, forming the sides of three small rectangles, represent uniform conducting wires having resistance $r$ per unit length, and current $I$ enters the mesh at $A$ and leaves at $C$. Show that the equivalent resistance of the circuit is

$$r(6\lambda^2 + 8\lambda + 1)/4(\lambda + 1)$$

If $\lambda = 1$, $r = 2$ ohms, $I = 5$ amperes, calculate the power put into the circuits. (D.)

3. A uniform wire of resistance $5R$ is bent to form the sides of a closed regular pentagon, and each vertex is connected to the centre by a wire of resistance $r$. Find the equivalent resistance of the network to a current entering at one of the vertices and leaving at the centre, and verify that it reduces to $5R/11$ when $r = R$. (R.)

4. Electric current enters a network of conducting wires at a given point and leaves it at a second given point. The system contains no batteries and the flow is steady. Show that of all the possible distributions of current which satisfy Kirchhoff's first law, and have the same inflow, the one that satisfies Kirchhoff's second law gives a minimum rate of heat production.

A pentagon $ABCDE$ is formed of fives wires each of resistance $r$. Each of the five vertices is joined to a point $O$ by a wire of resistance $kr$. Current enters the system at $A$ and leaves at $O$, these points being maintained at a constant potential difference. Show that the heat produced by the system is the same as that produced by a wire of resistance $kr(k^2 + 3k + 1)/(5k^2 + 5k + 1)$ whose ends are maintained at the same potential difference. (H.)

5. Two long, straight, uniform parallel wires are joined at equal intervals by cross-wires of length $a$, forming equal rectangles of sides $a$, $2a$. All the wires have the same resistance per unit length. A current enters and leaves the network at the ends $A$, $B$ of one of the cross-wires. Show that the equivalent resistance of the network is $r/\sqrt{2}$, where $r$ is the resistance of one of the cross-wires. (L.)

6. State Kirchhoff's Laws controlling the flow of current in a network of wires.

A telegraph line $A_0A_1 \ldots A_nA_{n+1}$ has $n+1$ identical sections of constant resistance $r$. The end $A_0$ is raised to a higher potential than that of $A_{n+1}$, which is earthed, the current entering the line at $A_0$ being $I_1$. At each point $A_s$ $(s = 1, 2, \ldots n)$ there is a leakage to earth of constant resistance $R$, the current leaving the line at $A_{n+1}$ being $I_{n+1}$. Prove that the current in the section $A_{s-1}A_s$ is

$$\frac{I_1 \sinh (n+1-s)\alpha + I_{n+1} \sinh (s-1)\alpha}{\sinh n\alpha}$$

where $1 + \dfrac{r}{2R} = \cosh \alpha$, and show that

$$I_1 \cosh \tfrac{1}{2}\alpha = I_{n+1} \cosh (n+\tfrac{1}{2})\alpha \qquad \text{(N.)}$$

7. A plane network is constructed of a regular polygon $A_1A_2 \ldots A_n$ formed from $n$ pieces of uniform wire, together with $n$ straight pieces of the same wire joining the vertices $A_1, A_2, \ldots A_n$ to the centre $O$ of the polygon.

A steady current $I$ enters the network at $A_1$ and leaves at $O$. Prove that, if $x_1, x_2, \ldots x_n$ denote the currents in the wires $A_1A_2, A_2A_3, \ldots A_nA_1$ respectively then

$$x_{s+2} - 2x_{s+1} \cosh 2\alpha + x_s = 0 \quad (s = 1, 2, \ldots n-2)$$

where $\cosh 2\alpha = 1 + \sin (\pi/n)$.

Verify that *all* the conditions of the problem are satisfied by

$$x_s = \frac{I}{2 \sinh n\alpha \cosh \alpha} \sinh (n+1-2s)\alpha \quad (s = 1, 2, \ldots n)$$

Prove also that the equivalent resistance of the network is $r \coth n\alpha \tanh \alpha$, where $r$ is the resistance of the wire $OA_1$. (H.)

8. A condenser of capacity $C$ carries a charge $Q_0$. At time $t = 0$ the plates are connected by a wire of resistance $R$ and inductance $L$. If $CR^2 < 4L$, show that at time $t$ the current in the circuit is

$$\frac{Q_0}{nLC} e^{-\sigma t} \sin nt$$

where $$\sigma = \frac{R}{2L}, \; n^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$$

Verify that the total heat developed in the circuit is equal to the original electrostatic energy of the condenser. (L.)

9. An open circuit consists of a coil of resistance $R_1$ and inductance $L_1$ and a battery of electromotive force $E$ connected in series. A second closed circuit of resistance $R_2$ and inductance $L_2$ lies near by. Show that if the first circuit is closed the total amount of charge which flows past any point of the second is $EM/R_1R_2$, where $M$ is the mutual inductance of the coil and the secondary circuit. (E.)

10. A simple circuit whose resistance is $R$ and inductance $L$ contains a condenser of capacity $C$. Show that if an alternating e.m.f. of frequency $\dfrac{2\pi}{p}$ is applied to the circuit the phase difference between the current and the applied e.m.f. is $\tan^{-1}\left\{\left(pL - \dfrac{1}{pC}\right)/R\right\}$ after a sufficient time has elapsed.

Obtain also an expression for the amplitude of the current flowing in the circuit. (E.)

11. State Kirchhoff's laws for a network whose members contain resistance, capacitance, and inductance and which is connected to a source of alternating e.m.f. Derive the rules for combining the complex impedances in series and in parallel.

In the network shown in the figure, $L$ and $R$ represent an inductance and a resistance respectively, and $C_1$, $C_2$ are capacitances. If $L < R^2C_2$, show that the network acts as a pure resistance for a current of frequency $\frac{1}{2\pi}\left[\frac{C_2 - L/R^2}{LC_2(C_1 + C_2)}\right]$.

(L.)

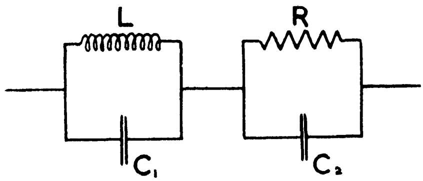


Fig. 37

12. Calculate the current in a circuit of resistance $R$ and inductance $L$ which contains a battery of electromotive $E_0 \cos pt$.

Prove that the introduction of a secondary circuit of inductance $N$ and resistance $S$ increases the effective resistance by

$$SMp^2/(S^2 + N^2p^2)$$

and diminishes the inductance by

$$NM^2p^2/(S^2 + N^2p^2)$$

where $M$ is the mutual inductance. (L.)

# CHAPTER 5
# ELECTROMAGNETISM

## 5.1 The Magnetic Field Due to a Current

If a current $i$ flows in a closed loop $C$ it produces a magnetic field, intensity $\mathbf{H} = -\text{grad}\,\phi$, where the value of $\phi$ at any point $P$ is given by

$$\phi_P = i\omega_P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$\omega_P$ being the solid angle subtended at $P$ by $C$ (but see 5.1.1).

$\phi$ is the usual magnetic potential defined, as in 2.1.2 (2), as the work done on a unit pole in going from $P$ to a fixed point $Q$, at zero potential.

**5.1.1 Ampere's Circuital Rule.** The magnetic (scalar) potential $\phi$ is not unique. Ampere's rule states that if a unit magnetic pole is taken on a closed path threading the loop $C$ positively $n$ times, an amount of work $4n\pi i$ is done by the field forces. The positive direction of threading $C$ is related to the direction of the current in $C$ by the usual right-handed screw rule.

It follows that $\phi$ is determinate only to within an integral multiple of $4\pi i$.

### 5.1.2 Simple Special Cases.

1. *A steady current i in an infinitely long straight wire.* It is easy to show that $\phi = -2i\theta$ and that $H_r = 0$, $H_\theta = \dfrac{2i}{r}$, $H_z = 0$, using cylindrical polar coordinates with $z$-axis along the wire.

2. *A steady current i in a circle, radius a.* From 5.1 (1) it is easy to deduce that at a point on the axis of the circle distant $z$ from the centre

$$\phi = 2\pi i\left(1 - \frac{z}{\sqrt{a^2 + z^2}}\right)$$

and the only non-zero component of intensity is

$$H_z = \frac{2\pi i a^2}{(z^2 + a^2)^{3/2}}$$

3. From (2) may be deduced the well-known result that the field inside a long solenoid with $n$ turns is $4\pi n i$.

## 5.2 Volume Currents. Vector Potential

Since there are no isolated magnetic poles, it is always true that

$$\text{div}\,\mathbf{H} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Also in any region where no current flows it may be shown that

$$\text{curl}\,\mathbf{H} = 0 \qquad \ldots \ldots \quad (3)$$

If, however, there is a volume current, current density **i**, (3) must be replaced by

$$\text{curl}\,\mathbf{H} = 4\pi\mathbf{i} \qquad \ldots \ldots \quad (4)$$

This implies that **H** can no longer be expressed as the gradient of a scalar function, but, as (2) still holds, we may write

$$\mathbf{H} = \text{curl}\,\mathbf{A} \qquad \ldots \ldots \quad (5)$$

where **A**, the *vector potential*, may be replaced by $\mathbf{A} + \text{grad}\,\phi$, $\phi$ being any scalar function of position (as curl grad $\phi \equiv 0$, 1.2).
To determine **A** a further condition

$$\text{div}\,\mathbf{A} = 0 \qquad \ldots \ldots \quad (6)$$

is usually imposed.
Equations (4), (5), (6) now lead to

$$\nabla^2\mathbf{A} = -4\pi\mathbf{i} \qquad \ldots \ldots \quad (7)$$

It may be shown that if **A** tends to zero at infinity and there are no current sheets (surface currents) a solution of (6) and (7) is

$$\mathbf{A} = \int \frac{\mathbf{i}\,dv}{r} \qquad \ldots \ldots \quad (8)$$

**5.2.1 The Biot–Savart Law.** If a current $i$ flows in a thin wire, the current density $\mathbf{i} = i\,d\mathbf{s}$, $d\mathbf{s}$ being a vector element of length, and (8) becomes

$$\mathbf{A} = i\int \frac{d\mathbf{s}}{r} \qquad \ldots \ldots \quad (9)$$

In particular, if a current $i$ flows in an infinite straight wire the vector potential is $-2i\mathbf{k}\log r$, where $r$ is the distance from the wire and **k** is a unit vector along the wire in the sense of the current.

From (9) it may be deduced that the magnetic field **H** at the field point $P$ is given by

$$\mathbf{H} = i\int \frac{d\mathbf{s} \times \mathbf{r}}{r^3} \qquad \ldots \ldots \quad (10)$$

**r** being from $d\mathbf{s}$ to $P$.

If $\theta$ is the angle between **r** and $d\mathbf{s}$ (10) expresses the fact that each element of the wire contributes a component $i \sin\theta\, ds/r^2$ to the magnetic intensity at $P$, in a direction perpendicular to the plane of $P$ and the element. This is the *Biot–Savart Law*.

(*Note*: In a medium of uniform permeability $\mu$ the right-hand sides of (9) and (10) must be multiplied by $\mu$, **B** replacing **H** in (10).)

### 5.3 Force on a Circuit in a Magnetic Field

If a closed circuit carrying a current $i$ is placed in a magnetic medium in which there is a magnetic field $\mathbf{H}$ (not including the field of the current itself), it may be shown that the potential energy of the system is

$$W = -i\int B_n\, dS = -iN \quad . \quad . \quad . \quad . \quad (11)$$

$N$ being the flux of magnetic induction, where the integral is taken over any surface with the circuit as boundary. From (11) it may be deduced that an element $d\mathbf{s}$ of the circuit experiences a force $iB \sin \theta\, ds$, ($\theta$ being the angle between $d\mathbf{s}$ and $\mathbf{B}$), in a direction perpendicular to the element and the induction and in the sense of a right-handed screw from the direction of the current to that of the induction. We can write

$$\mathbf{F} = i\, d\mathbf{s} \times \mathbf{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This is *Ampere's Law*.

**5.3.1** If a point charge $e$ is moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{H}$ and an electrostatic field $\mathbf{E}$ the resultant force on the charge is given by

$$\mathbf{F} = e\mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{B} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where $c$ is the ratio of the electro-magnetic to electro-static units of charge.

### 5.4 Coefficients of Mutual and Self-inductance

If currents flow in a number of circuits there is induced in each circuit an e.m.f. which is proportional to the rate of increase of $N$, the flux of magnetic induction, and is negative relative to the sense of the flow of induction.

If there is only one circuit with current $i$ we may write the induced e.m.f. as

$$E = -\frac{dN}{dt} = -L\frac{di}{dt} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where $N$ is the flux of induction, $L$ is a quantity, called the coefficient of self-inductance (or induction), which depends only on the size and shape of the circuit.

If there are two circuits with currents $i_1$, $i_2$ there will be an e.m.f. induced in each circuit by the current in the other (in addition to the e.m.f. due to its self-inductance, given by (14)). In the first circuit this will be

$$E_1 = -M\frac{di_2}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and similarly for $E_2$, where $M$ is the *coefficient of mutual inductance* of the two circuits and depends on their shapes, sizes and relative positions.

If $d\mathbf{s}_1$, $d\mathbf{s}_2$ are vector elements of the two circuits and $r$ the distance between them, then

$$M = \mu\oint\oint\frac{d\mathbf{s}_1 \, . \, d\mathbf{s}_2}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

(Neumann's formula).

If there are $n$ circuits $C_1, \ldots C_n$, carrying currents $i_1, \ldots i_n$, with self-inductances $L_1, \ldots L_n$ and mutual inductance $M_{rs}$ between $C_r$ and $C_s$ the magnetic energy of the currents is

$$\tfrac{1}{2}\Sigma L_s i_s^2 + \Sigma M_{rs} i_r i_s$$

## 5.5 Magnetic Vector Potential

The concept of a vector potential may be used in magnetostatics where we determine the vector $\mathbf{A}$ by the equations

$$\text{curl } \mathbf{A} = \mathbf{B}, \ \text{div } \mathbf{A} = 0$$

It may be shown that the magnetic vector potential for a magnetic dipole, moment $\mathbf{M}$, is

$$\mathbf{A} = \mathbf{M} \times \text{grad}\left(\frac{1}{r}\right) = (\mathbf{M} \times \mathbf{r})/r^3 \quad . \quad . \quad . \quad (17)$$

In general, if $\mathbf{I}$ is the intensity of magnetisation of an element of volume $dv$ and $r$ the distance of a field point $P$ from $dv$, then

$$\mathbf{A} = \int_v \frac{\text{curl } \mathbf{I} \, dv}{r} + \int_S \frac{\mathbf{I} \times \mathbf{n} \, dS}{r} \quad . \quad . \quad . \quad (18)$$

where $S$ is the surface enclosing the volume $v$. At a surface of separation of two magnetic media 1 and 2 the boundary conditions are

$$\mathbf{A}_1 = \mathbf{A}_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

(i.e. the vector potential is continuous) and

$$\frac{d\mathbf{A}_2}{dn} - \frac{d\mathbf{A}_1}{dn} = 4\pi(\mathbf{I}_2 - \mathbf{I}_1) \times \mathbf{n} \quad . \quad . \quad . \quad (20)$$

where $n$ is a unit vector normal to the surface of separation and drawn from medium 1 to medium 2.

## 5.6 Maxwell's Equations

In the notation of Chapter 2 these may be written in vector form as

$$\text{div } \mathbf{D} = 4\pi\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$\text{div } \mathbf{B} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$$\text{curl } \mathbf{H} = 4\pi\mathbf{i} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$\text{curl } \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

$$\mathbf{D} = K\mathbf{E} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$\mathbf{B} = \mu\mathbf{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$\mathbf{i} = \sigma\mathbf{E} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

**5.6.1 Boundary Conditions.** We suppose two media to be separated by a thin sheet which may carry a current (density vector **i**) and an electric charge (density $\sigma$). Take a unit vector **n** in the direction of the normal from medium 1 into medium 2.

Then it may be shown that the tangential components of **E**, **H**, and **i** are continuous across the surface of separation. The normal component of **B** is continuous. The normal component of **D** changes by $4\pi\sigma$. The condition for **H** is

$$\mathbf{H}_2 - \mathbf{H}_1 = 4\pi\mathbf{i} \times \mathbf{n}$$

If there is no current or residual charge in the surface of separation, then conditions reduce to:

$$D_n,\ B_n,\ E_t, \text{ and } H_t \text{ are continuous} \quad . \quad . \quad . \quad (28)$$

where $n$ denotes a normal and $t$ a tangential resolute.

**5.6.2 Energy.** 1. The energy per unit volume of an electromagnetic medium is $(K\mathbf{E}^2 + \mu\mathbf{H}^2)/8\pi$

2. The vector $\mathbf{S} = \dfrac{c}{4\pi}(\mathbf{E} \times \mathbf{H})$ (Poynting's Vector) measures the rate of flow of energy across unit area perpendicular to it.

## 5.7 Electromagnetic Waves

**5.7.1 Electromagnetic Waves in an Isotropic Dielectric.** In this case we put $\sigma$ and $\rho$ equal to zero and regard $K$ and $\mu$ as constant.

Equations (21)–(27) then reduce to

$$\text{div}\,\mathbf{E} = 0 = \text{div}\,\mathbf{H}$$

$$\text{curl}\,\mathbf{E} = -\frac{\mu}{c}\frac{\partial\mathbf{H}}{\partial t};\ \text{curl}\,\mathbf{H} = \frac{K}{c}\frac{\partial\mathbf{E}}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad (29)$$

It is easy to deduce that

$$\nabla^2\mathbf{E} = \frac{K\mu}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2};\ \nabla^2\mathbf{H} = \frac{K\mu}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2} \quad . \quad . \quad . \quad (30)$$

This is the standard wave equation, and the velocity of propagation for plane waves is known to be

$$v = c/\sqrt{K\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

*In vacuo* $K = 1 = \mu$ whence $v = c$.

If $n$ is the refractive index of a medium it is known that $\dfrac{c}{v} = n = \sqrt{K\mu}$, and since $\mu \doteq 1$ for a transparent medium, $n \doteq \sqrt{K}$.

**5.7.2 Plane Waves.** Consider a plane harmonic wave travelling with

speed $v$ in the direction defined by the unit vector $\mathbf{n} = (l, m, n)$. Each component of $\mathbf{E}$ and $\mathbf{H}$ must be a function of $lx + my + nz - vt$, e.g.

$$E_x = A \exp \{ip(lx + my + nz - vt)\} = B \exp ip\{(\sqrt{K\mu}(lx + my + nz) - ct)\}$$

using (31), etc.

(It is convenient to adopt the device here of using the complex exponential; the *actual* component $E_x$ is, of course, the real, or imaginary part of this expression.)

It may be shown that

$$\mathbf{n} \times \mathbf{H} = -\frac{Kv}{c}\mathbf{E} = -\left(\frac{K}{\mu}\right)^{1/2}\mathbf{E} \quad . \quad . \quad . \quad (32)$$

and

$$\mathbf{n} \times \mathbf{E} = \frac{\mu v}{c}\mathbf{H} = \left(\frac{\mu}{K}\right)^{1/2}\mathbf{H} \quad . \quad . \quad . \quad . \quad (33)$$

It follows that the vectors $\mathbf{n}$, $\mathbf{E}$, $\mathbf{H}$ form a right-handed triad of vectors, $\mathbf{E}$ and $\mathbf{H}$ being at right angles to the direction of propagation. The plane of $\mathbf{n}$ and $\mathbf{H}$ is called the *plane of polarisation.*

As an example we may write down a possible set of components of $\mathbf{E}$ and $\mathbf{H}$ for a plane harmonic wave propagated in the positive direction of the $z$-axis:

$$E_x = E_z = 0;\ E_y = ae^{i\omega\left(t - \frac{z}{v}\right)}$$

$$H_y = H_z = 0;\ H_x = -\left(\frac{K}{\mu}\right)^{1/2} ae^{i\omega\left(t - \frac{z}{v}\right)}$$

where $a$ may be complex.

($H_x$, $H_y$, $H_z$ may be determined from equation (33).)

It is necessary only for the terms in $t$ and $z$ in the exponent to be in the ratio $v$, and so we can write $E_y$, for instance, as $a_1 e^{i\omega(vt-z)}$ or, using (31) $a_2 e^{i\omega\,(ct - \sqrt{K\mu}z)}$. It should be noted, as a consequence of (32) or (33), that $K^{1/2}\,|\mathbf{E}| = \mu^{1/2}\,|\mathbf{H}|$ whence

$$\frac{K\mathbf{E}^2}{8\pi} = \frac{\mu\mathbf{H}^2}{8\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

i.e. the electric and magnetic energies have the same density.

**5.7.3 Waves in a Conducting Medium.** Now $\sigma \neq 0$ and equations (21)-(27) give

$$\operatorname{div} \mathbf{E} = 0 = \operatorname{div} \mathbf{H}$$

$$\operatorname{curl} \mathbf{E} = \frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t},\ \operatorname{curl} \mathbf{H} = 4\pi\sigma\mathbf{E} + \frac{K}{c}\frac{\partial \mathbf{E}}{\partial t}$$

from which it may easily be shown that

$$\nabla^2\mathbf{E} = \frac{K\mu}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} + \frac{4\pi\sigma\mu}{c}\frac{\partial \mathbf{E}}{\partial t} \quad . \quad . \quad . \quad (35)$$

with a similar equation for $\mathbf{H}$ (the *equation of telegraphy*).

If we are dealing with waves of period $2\pi/\omega$, so that $E$ is proportional to $e^{i\omega t}$, we may write (35) in the form

$$\nabla^2\mathbf{E} = \frac{\mu}{c^2}\left(K - i\frac{4\pi\sigma}{\omega}\right)\frac{\partial^2\mathbf{E}}{\partial t^2} \qquad . \quad . \quad . \quad . \quad (36)$$

Comparing this with equation (30), we see that the propagation of waves in a conducting medium is the same as for an isotropic dielectric with a *complex* dielectric constant $K^1 = K - i\frac{4\pi\sigma}{\omega}$.

The analysis of 5.7.2 may now be carried through with $K^1$ in place of $K$.

## 5.8 Scalar and Vector Potentials in Maxwell's Equations

An alternative approach to the solution of Maxwell's equations (21)–(27) is as follows:

Since div $\mathbf{B} = 0$ there is a vector $\mathbf{A}$ such that

$$\mathbf{B} = \text{curl } \mathbf{A} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Moreover, $\mathbf{A}$ is indeterminate (cf. 5.1.2), and we may choose div $\mathbf{A}$ at will.

Substitution in (24) gives

$$\text{curl}\left(\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right) = 0$$

whence there exists a scalar $\phi$ such that

$$\mathbf{E} = -\text{grad } \phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} \qquad . \quad . \quad . \quad . \quad . \quad (38)$$

If we impose the condition

$$\text{div } \mathbf{A} + \frac{K\mu}{c}\frac{\partial\phi}{\partial t} = 0 \qquad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

it may be shown that

$$\nabla^2\phi - \frac{K\mu}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{4\pi\rho}{K} \qquad . \quad . \quad . \quad . \quad . \quad (40)$$

$$\nabla^2\mathbf{A} - \frac{K\mu}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -4\pi\mu\mathbf{i} \qquad . \quad . \quad . \quad . \quad . \quad (41)$$

and

$$\text{div } \mathbf{i} + \frac{1}{c}\frac{\partial\rho}{\partial t} = 0 \qquad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

If we can solve equations (40) and (41), subject to (42), a solution of Maxwell's equations is given by

$$\mathbf{B} = \text{curl } \mathbf{A}, \ \mathbf{E} = -\text{grad } \phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$$

$\phi$ and $\mathbf{A}$ are called the electromagnetic potentials. Clearly if $\phi_0$ and $\mathbf{A}_0$ are solutions of (39)–(41), then

$\phi = \phi_0 + \frac{1}{c}\frac{\partial\psi}{\partial t}$ and $\mathbf{A} = \mathbf{A}_0 - \text{grad}\,\psi$ are also solutions provided that

$$\nabla^2\psi = \frac{K\mu}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

**5.8.1 Retarded Potentials.** It can also be shown, by a method of integration due to Kirchhoff, that a solution of (40) and (41) is given by

$$\phi = \frac{1}{K}\int\frac{[\rho]}{r}\,dv \quad . \quad . \quad . \quad . \quad . \quad (43)$$

and

$$\mathbf{A} = \mu\int\frac{[\mathbf{i}]}{r}\,dv \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

where $[\rho]$ and $[\mathbf{i}]$ are the values of $\rho$, $\mathbf{i}$ at time $t - r/c$, $r$ being the distance from the charge or current element.

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**Problem 56**

A circular loop of wire, containing a periodic electromotive force of constant amplitude $E$ and period $2\pi/\omega$, rotates with uniform angular velocity $\omega$ about a fixed vertical diameter. The radius of the loop is $a$, the resistance of the wire is $R$ and its self-inductance is $L$. If $H$ is the horizontal component of the earth's magnetic field, prove that the current $i$ induced in the wire satisfies an equation of the form

$$\frac{Ldi}{dt} + Ri = E\cos(\omega t + \alpha) - \pi a^2 H\omega\cos\omega t$$

Show that for variable $\alpha$ the periodic component of the induced current has a maximum amplitude

$$(E + \pi a^2 H\omega)/(R^2 + L^2\omega^2)^{1/2}$$

and that in this case it differs in phase from the electromotive force by $\tan^{-1}(L\omega/R)$. (H.)

---

**Solution.**

When the horizontal diameter of the loop makes an angle $\omega t$ with the earth's horizontal magnetic field $H$, the flux through the loop is $\pi a^2 H\sin\omega t$ and so the induced back e.m.f. in the loop is $d/dt\,(\pi a^2 H\sin\omega t)$, i.e. is $\pi a^2 H\omega\cos\omega t$.

The forward e.m.f. in the loop is given to have period $2\pi/\omega$, amplitude $E$, and thus is $E\cos(\omega t + \alpha)$, the $\alpha$ giving a phase difference between the two periodic changes, in position and in e.m.f.

The equation satisfied by the current in the loop is then, as in 4.1.1 or 5.1.2.3

$$L\frac{di}{dt} + Ri = E\cos(\omega t + \alpha) - \pi a^2 H\omega \cos \omega t \quad . \quad (1)$$

The periodic part of the induced current is the steady state solution of this, i.e.

$$\frac{E}{(R^2 + L^2\omega^2)^{1/2}}\cos(\omega t + \alpha - \beta) - \frac{\pi a^2 H\omega}{(R^2 + L^2\omega^2)^{1/2}}\cos(\omega t - \beta) \quad . \quad (2)$$

where $\tan\beta = L\omega/R$.

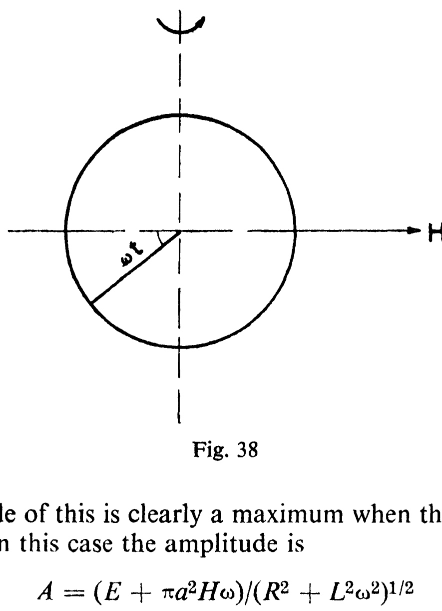


Fig. 38

The amplitude of this is clearly a maximum when the terms add, i.e when $\alpha = \pi$. In this case the amplitude is

$$A = (E + \pi a^2 H\omega)/(R^2 + L^2\omega^2)^{1/2}$$

and the induced current is $-A\cos(\omega t - \beta)$, and so differs in phase from the e.m.f. $E\cos(\omega t + \alpha)$, i.e. $-E\cos\omega t$ (since $\alpha = \pi$) by $\beta = \tan^{-1} L\omega/R$.

---

**Problem 57**

A current $2I$ flows in a long straight wire whose position referred to a right-handed rectangular system of axes is $x = 0$, $z = a$, and a current $I$ flows in another wire given by $y = 0$, $z = -a$, the senses of the currents being respectively those of $y$ and $x$ increasing. A small magnet of moment $M$, pivoted at the origin of coordinates, is free to move in any direction about its centre. Determine the positions of equilibrium and discuss their stability.

Prove also that the mechanical force acting on the pivot in either position of equilibrium is $6MI/(a^2\sqrt{5})$. (L.)

---

**Solution.**

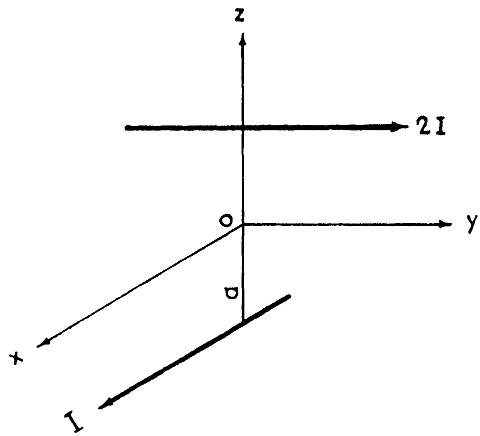


Fig. 39

The magnetic field due to the current $2I$ is, from 5.1.1.1 (1), at $O$,

$$\left(\frac{-4I}{a}, 0, 0\right) \text{ and, near } O, \text{ at } (x, y, 0) \text{ is } \left(\frac{-4Ia}{a^2+x^2}, 0, \frac{-4Ix}{a^2+x^2}\right)$$

The magnetic field due to the current $I$ is, at $O$, $\left(0, \frac{-2I}{a}, 0\right)$ and, near $O$, at $(x, y, 0)$ is $\left(0, \frac{-2Ia}{a^2+y^2}, \frac{2Iy}{a^2+y^2}\right)$

A small magnet of moment $M$ at $O$ therefore is in equilibrium when it lies in the direction of the resultant field, i.e. when it lies along the line with direction cosines $(2/\sqrt{5}, 1/\sqrt{5}, 0)$.

The potential energy of a magnet $\mathbf{M}$ in a field $\mathbf{H}$ is $-\mathbf{M}\,.\,\mathbf{H}$ and so is a minimum, i.e. the equilibrium is *stable*, if $\mathbf{M}$ and $\mathbf{H}$ are in the *same* direction, unstable if $\mathbf{M}$ and $\mathbf{H}$ are in *opposite* directions.

The moment $\mathbf{M}$ is $(\mp 2/\sqrt{5}M, \mp 1/\sqrt{5}M, 0)$ in the equilibrium states, and so the potential energy $W = -\mathbf{M}\,.\,\mathbf{H}$ and the force on the magnet is $\mathbf{F} = (\mathbf{M}\,.\,\text{grad})\,\mathbf{H}$

$$= \left(M_1\frac{\partial}{\partial x} + M_2\frac{\partial}{\partial y}\right)\left(\frac{-4Ia}{a^2+x^2}, \frac{-2Ia}{a^2+y^2}, \frac{-4Ix}{a^2+x^2} + \frac{2Iy}{a^2+y^2}\right)$$

which becomes on expansion

$$\mp\frac{2}{\sqrt{5}}\frac{8MIax}{(a^2+x^2)^2},\ \mp\frac{1}{\sqrt{5}}\frac{4MIay}{(a^2+y^2)^2},$$

$$\pm\frac{2}{\sqrt{5}}M\left\{\frac{Ix^2}{(a^2+x^2)^2}-\frac{4I}{a^2+x^2}\right\}\mp\frac{1}{\sqrt{5}}M\left\{\frac{-4Ix^2}{(a^2+y^2)^2}+\frac{2I}{a^2+y^2}\right\}$$

and, at $x = 0, y = 0$, is $(0, 0, \pm 6IM/\sqrt{5}a^2)$.

Thus the force on the pivot in either position of equilibrium is in the $z$-direction, and is of magnitude $6IM/\sqrt{5}a^2$, upwards when the equilibrium is stable, downwards when it is unstable.

## Problem 58

A current $i$ flows in a thin wire, forming a circuit $C$, in the presence of an external magnetic field of intensity $\mathbf{H}$. Prove, by considering the equivalent magnetic shell, or otherwise, that the mechanical force $\mathbf{F}$ acting on the circuit is $\mathbf{F} = i\oint d\mathbf{s} \times \mathbf{H}$ (e.m. units), where $d\mathbf{s}$ is an element of $C$.

A current $i$ flows in a circular wire of radius $a$, and a current $i'$ flows in an infinite straight wire in the same plane and at a distance $b$ ($>a$) from the centre of the circle. Show that the force between the wires is

$$4\pi ii'\{(1 - a^2/b^2)^{-1/2} - 1\}$$

Indicate the senses of the currents if the force is one of attraction.

You may assume that

$$\int_0^{2\pi}\frac{d\theta}{1+\alpha\cos\theta} = \frac{2\pi}{\sqrt{(1-\alpha^2)}},\ (-1 < \alpha < 1) \qquad \text{(L.)}$$

**Solution.**

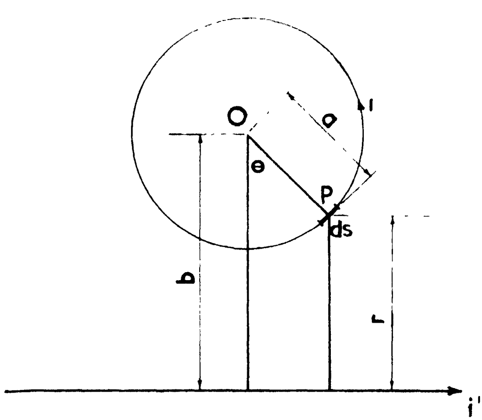


Fig. 40

The magnetic field **H** at $P$ due to the current $i'$ is $2i'/r$ right-hand screw positive with respect to $i'$, normal to the plane of the circle and wire; and the element at $P$ of the circuit carrying current $i$ is $d\mathbf{s}$. Thus by the result in the first part of the question (Ampere's Law), the force $d\mathbf{F}$ on the element at $P$ is

$$i\,d\mathbf{s} \times \mathbf{H} = 2\,ii'/r\,ds \text{ along the radius } OP$$

By symmetry the total force $F$ on the circle is along $OY$ and so, resolving in this direction,

$$F = \int_0^{2\pi} \frac{2ii'a}{r} \cos\theta\, d\theta$$

with $$r = b - a\cos\theta,$$

so that $$F = 2ii' \int_0^{2\pi} \frac{a\cos\theta\, d\theta}{b - a\cos\theta}$$

$$= 2ii'\left\{-2\pi + \int_0^{2\pi} \frac{d\theta}{1 - a/b\cos\theta}\right\}$$

$$= 2ii'\{-2\pi + 2\pi/\sqrt{1 - a^2/b^2}\} \text{ by the result given,}$$

i.e. $$F = 4\pi ii'\{(1 - a^2/b^2)^{-1/2} - 1\} \text{ as required.}$$

This is positive, since $(1 - a^2/b^2)^{-1/2} > 1$, and so these directions of the two currents give an attraction between them.

*Alternatively*, the force may be obtained by considering the potential energy of the circular current. This is, from 5.3 (11)

$$W = -iN \text{ where } N = \int H\, dA$$

$H$ being the normal component of the field due to current $i'$ and $dA$ the element of area of the circle. Here the field is wholly normal, so that

$$H = 2i'/r,\ r = b - a\cos\theta \text{ and } dA = 2a^2\sin^2\theta\, d\theta$$

Then $$W = -4ii'a^2 \int_0^{\pi} \frac{\sin^2\theta\, d\theta}{b - a\cos\theta}$$

$$= -4ii' \int_0^{\pi} \left\{\frac{a^2 - b^2}{b - a\cos\theta} + b + a\cos\theta\right\} d\theta$$

$$= -4ii'\left\{\left(\frac{a^2 - b^2}{b}\right)\frac{\pi}{\sqrt{1 - a^2/b^2}} + b\pi\right\} \text{ using the given result}$$

$$= -4ii'\{b - \sqrt{b^2 - a^2}\}$$

and so the force of attraction $= \dfrac{\partial W}{\partial b} = 4\pi ii'\{(1 - a^2/b^2)^{-1/2} - 1\}$ as required.

**Problem 59**

By considering the equivalent magnetic shell, or otherwise, prove that, in the usual notation, the coefficient of mutual induction between closed circuits is given by

$$\oint\oint \frac{d\mathbf{s}_1 \cdot d\mathbf{s}_2}{\mathbf{r}_1 - \mathbf{r}_2}$$

Two parallel fixed wires given by $x = \pm a$, $y = 0$, carry fixed currents $\pm I$. Show that the electromotive force per unit length induced in the parallel moving wire $x = x(t)$, $y = y(t)$, is given by

$$2I\left\{\frac{\dot{x}(x-a)+\dot{y}y}{(x-a)^2+y^2} - \frac{\dot{x}(x+a)+\dot{y}y}{(x+a)^2+y^2}\right\} \qquad \text{(L.)}$$

---

**Solution.**

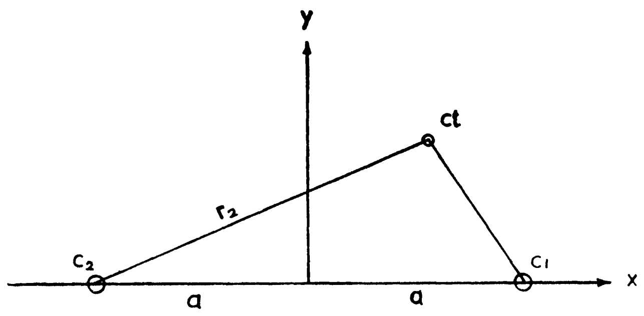


Fig. 41

The induced e.m.f. in a circuit is —(rate of change of flux of magnetic induction) (5.4).

If $L_{12}$ is the mutual induction of circuits $C_1$ and $C_2$, then the flux through $C_1$ due to current $I_2$ in $C_2$ is $L_{12}I_2$.

Thus, if $C_1$ be the circuit composed of the wire through $(a, 0)$ carrying current $I$ and the wire through $(-a, 0)$ carrying current $-I$ and $C_t$ be unit length of the wire through $(x, y)$, then the flux through $C_t$ is $L_{1t}I$ (since $C_t$ has no self-induction), and the induced e.m.f. is

$$-\left(I\frac{dL_{1t}}{dt}\right)$$

Using the formula,

$$L_{1t} = \int_0^1 ds_t \int_{-\infty}^{\infty} ds_1 \left\{\frac{1}{\sqrt{(r_1^2 + s_1^2)}} - \frac{1}{\sqrt{(r_2^2 + s_1^2)}}\right\}$$

$$\text{Now} \int_{-S}^{S} \left\{ \frac{1}{\sqrt{(r_1^2 + s_1^2)}} - \frac{1}{\sqrt{(r_2^2 + s_1^2)}} \right\} ds_1$$

$$= \log \left\{ \frac{S + \sqrt{r_1^2 + S^2}}{-S + \sqrt{(r_1^2 + S^2)}} \right\} \left\{ \frac{-S + \sqrt{(r_2^2 + S^2)}}{S + \sqrt{(r^2_2 + S^2)}} \right\}$$

$$= \log \left\{ \frac{S + \sqrt{r^2 + S^2}}{S + \sqrt{r_2^2 + S^2}} \right\} \left( \frac{r_2^2}{r_1^2} \right) \text{ expanding since } S \text{ is large}$$

$$\longrightarrow 2 \log (r_2/r_1) \text{ as } S \longrightarrow \infty$$

$$\text{Thus } L_{1t} = 2 \int_0^1 ds_t \log (r_2/r_1)$$

$$= 2 \log (r_2/r_1) = 2 \log \frac{\{(x + a)^2 + y^2\}^{1/2}}{\{(x - a)^2 + y^2\}^{1/2}}$$

$$= \log \{(x + a)^2 + y^2\} - \log \{(x - a)^2 + y^2\}$$

and the induced e.m.f. is

$$-\frac{IdL_{1t}}{dt} = 2I \left\{ \frac{\dot{x}(x - a) + \dot{y}y}{(x - a)^2 + y^2} - \frac{\dot{x}(x + a) + \dot{y}y}{(x + a)^2 + y^2} \right\}$$

as required.

---

**Problem 60**

Show that, in a cylindrical problem in which the field is due to currents in straight wires parallel to the $z$-axis,

$$B_x = \partial A_z/\partial y \text{ and } B_y = -\partial A_z/\partial x$$

where $\mathbf{A}$ denotes the vector potential. (The result $\mathbf{A} = \mu I \int r^{-1} d\mathbf{s}$ for the vector potential due to the current $I$ may be assumed.)

Hence show that, at the interface separating media of different permeability, $\partial A_z/\partial S$ and $\mu^{-1}\partial A_z/\partial n$ are continuous, where $\partial S$ and $\partial n$ denote elements in the directions of the tangent and normal to the section of the interface.

A current $I$ flows along the wire parallel to the plane face of a semi infinite medium of permeability $\mu$ and distant $d$ from it. By considering a corresponding potential problem with a dielectric, or otherwise, show that the force per unit length on the wire is an attraction

$$\left( \frac{\mu - 1}{\mu + 1} \right) I^2/d \qquad \text{(O.)}$$

---

**Solution**

The vector potential $\mathbf{A}$ due to current $I$ in a circuit element $d\mathbf{s}$ is (from 5.1.2 (8)) $\mu I/r d\mathbf{s}$; and the vector potential $\mathbf{A}$ due to currents in straight

wires parallel to the $z$-axis is thus in the $z$-direction only, i.e. is $(0, 0, A_z)$. Hence, since $\mathbf{B} = \text{curl } \mathbf{A}$,

$$B_x = \frac{\partial A_z}{\partial y}, \; B_y = -\frac{\partial A_z}{\partial x} \text{ as required.}$$

To calculate conditions at an interface, take the $x$-axis tangential, the $y$-axis normal to the interface and suppose the transition between the two media to occur through a layer of thickness $\delta$, in which $\mu$ changes continuously from $\mu_1$ to $\mu_2$.

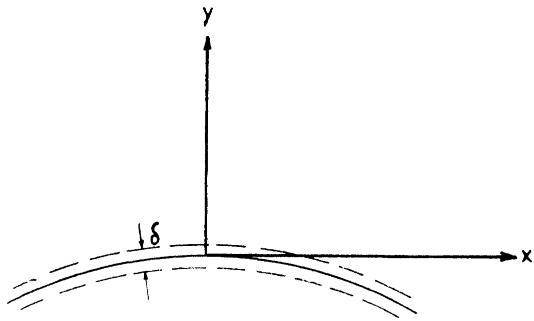


Fig. 42

Then the vector potential $\mathbf{A}$ is continuous at the surface,

i.e. $\quad A_z(0, \delta) = A_z(0, -\delta)$ and $A_z(\xi,\delta) = A_z(\xi, -\delta)$

and so $\left(\frac{\partial A_z}{\partial x}\right)$ at $x = \xi/2$, $y = \delta$, equals $\left(\frac{\partial A_z}{\partial x}\right)$ at $x = \xi/2, y = -\delta$, i.e.

$\frac{\partial A_z}{\partial s}$ is continuous, where $\partial s$ is the general tangential element (here $\partial s = \partial x$).

Also curl $\mathbf{H} = 4\pi\mathbf{i}$, where $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{i}$ is the current density i.e. within the transition layer, curl $\left(\frac{\mathbf{B}}{\mu}\right) = 4\pi\mathbf{i}$, whence

$$\mathbf{i} = \frac{1}{4\pi}\left\{-\frac{\partial}{\partial z}\left(\frac{B_y}{\mu}\right), \frac{\partial}{\partial z}\left(\frac{B_x}{\mu}\right), \frac{\partial}{\partial x}\left(\frac{B_y}{\mu}\right) - \frac{\partial}{\partial y}\left(\frac{B_x}{\mu}\right)\right\}$$

But the current density in the transition layer must have zero component in the normal ($y$) direction, i.e. $\frac{1}{4\pi}\frac{\partial}{\partial z}\left(\frac{B_x}{\mu}\right) = 0$ i.e. $\frac{B_x}{\mu}$ must be constant through the layer i.e. $\frac{1}{\mu}\frac{\partial A_z}{\partial n}$ must be continuous on both sides of the interface where $\partial n$ is the general normal element (here $\partial n = \partial y$).

Given a current flowing along an infinite wire parallel to the plane face of a medium, permeability $\mu$, the vector potential has only one

component $A_z$ and the conditions to be satisfied are that (i) $\frac{\partial A_z}{\partial s}$ is continuous, and (ii) $\frac{1}{\mu}\frac{\partial A_z}{\partial n}$ is continuous at the plane face. Also the vector potential due to the wire *alone* would be $A_z = -2I \log r$ in free space, $r$ being distance from the wire (from 5.2.1).

It follows that the conditions are identical with the corresponding potential problem of a line charge $2I$ in the presence of a plane dielectric $1/\mu$ where the potential $\phi$ has to satisfy $\phi = -2I \log r$ due to charge alone and $\frac{\partial \phi}{\partial s}, \frac{1}{\mu}\frac{\partial \phi}{\partial n}$ continuous at the plane face.

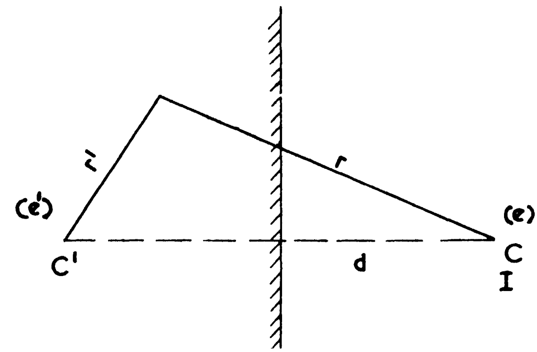


Fig. 43

To solve this, try image* line charges $e'$, $e$ at the image point $C^1$ (for free space) and at $C$ (for dielectric) and let

$$\phi_d = \phi \text{ in dielectric} = -e \log r$$
$$\phi_f = \phi \text{ in free space} = -e' \log r' - 2I \log r$$

Then we need $\phi_d = \phi_f$ on the plane $r = r'$, so that

$$-e = -e' - 2I \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and $\frac{1}{\mu}\frac{\partial \phi_d}{\partial n} = \frac{\partial \phi_f}{\partial n}$ on the plane, so that

$$\frac{1}{\mu}\frac{e}{r}\cdot\frac{d}{r} = -\frac{e'}{r'}\cdot\frac{d}{r'} + \frac{2I}{r}\frac{d}{r} \text{ on the plane}$$

i.e.
$$+e/\mu = -e' + 2I \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Solving (1) and (2) gives

$$e = 4\mu I/(\mu + 1),\ e' = 2I(\mu - 1)/(\mu + 1) \quad . \quad . \quad . \quad (3)$$

* The method of images is dealt with in Vol. II.

As in the corresponding potential problem, the force on the wire at $C$ is the same as if the material were removed and there were another wire at $C'$ carrying current $I' = e'/2$ in the same sense.

This force is $2II'/2d = \dfrac{I^2(\mu - 1)}{d(\mu + 1)}$ as required.

---

**Problem 61**

Write down Maxwell's equations for the electromagnetic field *in vacuo*. Hence prove that the rate of diminution of electromagnetic energy in a fixed volume equals the integral of Poynting's vector over the surface of the volume.

Given that an electric dipole of moment $\mathbf{M} = \mathbf{M}_0 \cos \omega t$ (where $\mathbf{M}_0$ is a constant vector) produces, at large distances, the field

$$\mathbf{E} = \frac{\omega^2}{c^2} \frac{\mathbf{r} \times (\mathbf{M}_0 \times \mathbf{r})}{|\mathbf{r}|^3} \cos \left\{\omega\left(t - \frac{|\mathbf{r}|}{c}\right)\right\}$$

$$\mathbf{H} = -\frac{\omega^2}{c^2} \frac{\mathbf{M}_0 \times \mathbf{r}}{|\mathbf{r}|^2} \cos \left\{\omega\left(t - \frac{|\mathbf{r}|}{c}\right)\right\}$$

show that the mean rate of radiation of energy is $\frac{1}{3}\, \mathbf{M}_0{}^2\, \frac{\omega^4}{c^3}$ (L.)

---

**Solution.**

By the first part, the mean rate of radiation of energy is the time mean of the integral of the Poynting vector $\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H}$ over the surface of a sphere.

Since $\quad \mathbf{E} = A\, \mathbf{r} \times (\mathbf{M}_0 \times \mathbf{r})$

and $\quad \mathbf{H} = B(\mathbf{M}_0 \times \mathbf{r})$

$$\begin{aligned}\mathbf{E} \times \mathbf{H} &= AB[\{(\mathbf{M}_0 \times \mathbf{r}) \,.\, \mathbf{r}\}\, (\mathbf{M}_0 \times \mathbf{r}) - (\mathbf{M}_0 \times \mathbf{r})^2 \mathbf{r}] \\ &= -AB(\mathbf{M}_0 \times \mathbf{r})^2 \mathbf{r}\end{aligned}$$

where $A = \dfrac{\omega^2}{c^2 r^2} \cos \{\omega(t - r/c)\}$, $B = -\dfrac{\omega^2}{c^2 r^2} \cos \{\omega(t - r/c)\}$

On the surface $r = R$, we have, measuring $\theta$ from the direction of $\mathbf{M}_0$, $(\mathbf{M}_0 \times \mathbf{r})^2 = \mathbf{M}_0{}^2 R^2 \sin^2 \theta$ and $\mathbf{r} = (R, 0, 0)$ in spherical polars. The integral of $\mathbf{E} \times \mathbf{H}$ over the surface is then

$$\int (\mathbf{E} \times \mathbf{H}) \,.\, \mathbf{n}\, dS = \int (\mathbf{E} \times \mathbf{H}) \,.\, \mathbf{r} \frac{dS}{r}$$

Thus the flux of the Poynting vector is

$$\frac{c}{4\pi}\frac{\omega^4}{c^4R^5}\cos^2\{\omega(t - R/c)\}\int_0^\pi M_0{}^2R^2\sin^2\theta\,.\,R\,.\,2\pi R^2\sin\theta\,d\theta$$

$$= \frac{2}{3}\frac{M_0{}^2\omega^4}{c^3}\cos^2\{\omega(t - R/c)\}$$

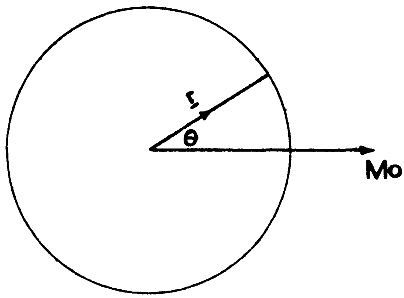


Fig. 44

and, since the average value of $\cos^2\theta$ is $\frac{1}{2}$, the time average of this flux is $\frac{1}{3}\frac{M_0{}^2\omega^4}{c^3}$, which is the mean rate of radiation of energy as required.

---

**Problem 62**

Show that the scalar potential $\phi$ and the vector potential **A** of a system of currents and charges in free space may be chosen to satisfy

$$\Box\phi = -\rho/t_0,\ \Box\mathbf{A} = -\mu_0\mathbf{j},\ \text{where } \Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

A constant current $J$ is started at $t = 0$ in an infinite straight conducting wire. Find the vector potential **A** at a point $P$ distant $r$ from the wire at time $t$, where $r < ct$. (M.)

---

**Solution.** The result in the first part of this question is an alternative form of equations 5.8 (40) and (41).

A solution of these equations is known to be, from 5.8.1 (43) and (44),

$$\phi = \frac{1}{K}\int\frac{[\rho]}{r}\,dv$$

$$\mathbf{A} = \mu\int\frac{[\mathbf{j}]}{r}\,dv$$

where $[\rho]$, $[\mathbf{j}]$ are the values of $\rho$, $\mathbf{j}$ the charge and current densities, at time $t - r/c$, and $r$ is the distance of the external point from the charge or current.

If a current $J$ is started in an infinite straight wire, in direction $\mathbf{k}$, then we have $\mathbf{j} = J\mathbf{k}$, $t \geqslant 0$; $\mathbf{j} = 0$; $t < 0$ and the potentials at an external point $P$, at distance $r$ from the wire, are given by

$$\phi = 0 \text{ (since } \rho = 0 \text{ everywhere) and}$$

$$\mathbf{A} = \mu \int_{-\infty}^{\infty} [j]/R \, dx$$

But

$$\begin{aligned} [\mathbf{j}] &= \mathbf{j} \text{ at time } t - R/c \\ &= 0 \text{ when } R \geqslant ct, \text{ i.e. when } |x|^2 \geqslant c^2t^2 - r^2 \\ &= J\mathbf{k} \text{ otherwise} \end{aligned}$$

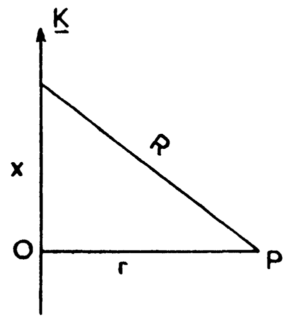


Fig. 45

Thus $\mathbf{A} = \mu J\mathbf{k} \displaystyle\int_{-\sqrt{c^2t^2-r^2}}^{+\sqrt{c^2t^2-r^2}} \frac{dx}{\sqrt{x^2+r^2}}$, $r < ct$, $\mathbf{A} = 0$ otherwise

i.e.

$$\begin{aligned} \mathbf{A} &= 2\mathbf{k}J\mu \left[\log \{x + \sqrt{x^2+r^2}\}\right]_0^{\sqrt{c^2t^2-r^2}} \\ &= 2\mathbf{k}J\mu \log \frac{\sqrt{c^2t^2-r^2}+ct}{r}, \quad r < ct; \ \mathbf{A} = 0, \ r \geqslant ct \end{aligned}$$

and this is the required vector potential.

---

**Problem 63**

Prove that, for any vector $\mathbf{F}$ having components with continuous first derivatives,

$$\int_v \operatorname{curl} \mathbf{F} \, dv = \int_S (\mathbf{n} \times \mathbf{F}) \, dS$$

where $S$ is a smooth closed surface bounding a volume $v$, and $\mathbf{n}$ is the unit vector in the direction of the outward drawn normal to $S$ at the element $dS$.

Assuming the formula $\mathbf{A} = (\mathbf{m} \times \mathbf{r})/r^3$ for the magnetic vector potential at the point $\mathbf{r}$ due to a dipole of moment $\mathbf{m}$ located at the origin of coordinates, prove that, at an external point $P$, the vector

potential of a finite body, magnetised with intensity **I** and bounded by a closed surface $S$, is

$$\mathbf{A} = \int_v \frac{\text{curl}\,\mathbf{I}}{r}\,dv + \int_S \frac{\mathbf{I}\times\mathbf{n}}{r}\,dS$$

where $r$ is the distance of the element $dv$ or $dS$ from $P$.

Deduce, or prove otherwise, that if the magnet is a uniformly magnetised sphere of radius $a$, then

$$\mathbf{A} = \frac{4\pi a^3\,(\mathbf{I}\times\mathbf{R})}{3R^3}$$

at an external point whose position vector is **R** relative to the centre of the sphere.

(The formula curl $(\phi\mathbf{F}) = \phi$ curl $\mathbf{F}$ + (grad $\phi$) $\times$ $\mathbf{F}$ may be assumed.)
(O.)

**Solution.** The first part is a standard vector result (1.3 ($a$) (iii)).

The second part is a standard result on magnetisation (5.4(18)).

For the final part, we have

$$\mathbf{A} = \int_v \frac{\text{curl}\,\mathbf{I}}{r}\,dv + \int_S \frac{\mathbf{I}\times\mathbf{n}}{r}\,dS$$

where the integrals are to refer to the interior and surface respectively of a uniformly magnetised sphere $r = a$. In this case curl $\mathbf{I} = 0$ in the interior of the sphere (since the magnetisation is uniform); so that

$$\mathbf{A} = \int_S \frac{\mathbf{I}\times\mathbf{n}}{r}\,dS$$

where $r$ is the distance of the element $dS$ of the sphere's surface from the field point $r = \mathbf{R}$.

Since **I** is a constant vector, $\mathbf{A} = \mathbf{I} \times \int \frac{\mathbf{n}\,dS}{r}$ and the integral must be a vector along the direction of **R**.

Hence integrating components in this direction gives

$$\mathbf{A} = \mathbf{I}\times\mathbf{R}\int_0^{\pi} \frac{2\pi a^2\cos\theta\sin\theta\,d\theta}{\{a^2 + R^2 - 2aR\cos\theta\}^{1/2}}$$

and this, on integrating, gives the required result.

*Alternatively*, we may regard a uniformly magnetised sphere as the superposition of two identical spheres, one made up of positive poles with density $\rho$ and the other of negative poles with the same density, displaced from each other by a small displacement $\delta$. Then the potential at an external point due to a uniform distribution through a sphere is the same as if the distribution were collected at the centre, and so the potential here is the same as for positive and negative poles of magnitude $\frac{4}{3}\pi a^3\rho$ at distance $\delta$, i.e. as for a dipole $\frac{4}{3}\pi a^3\rho\delta$, or $\frac{4}{3}\pi a^3$ **I**, since $\mathbf{I} = \rho\delta$,

the magnetisation per unit volume. (This has been proved for scalar potential, and since there is a unique vector potential corresponding to any given scalar potential, it holds also for vector potential.)

Hence, using the result given for a dipole, the vector potential at an external point **R** is

$$\tfrac{4}{3}\pi a^3\, \mathbf{I} \times \mathbf{R}/R^3 \text{ as required.}$$

---

**Problem 64**

Prove that outside a straight wire parallel to the $z$-axis and carrying a current of strength $j$ the vector potential is $-2j\mathbf{k} \log r$, where $r$ is the distance from the wire and $\mathbf{k}$ is a unit vector along the wire in the sense of the current.

If $n$ parallel straight wires each carrying a current of strength $j$ in the same sense are placed at equal intervals round the circle $r = a$, $z = 0$, and normal to its plane, prove that the vector potential at the point $P$ of cylindrical polar coordinates $(r, \theta, z)$ can be expressed in the form

$$\mathbf{A} = -j\mathbf{k} \log (r^{2n} + a^{2n} - 2r^n a^n \cos n\theta)$$

Hence, or otherwise, show that the mechanical force per unit length on one of the wires is $j^2(n - 1)/a$ towards the $z$-axis. (L.)

---

**Solution.**

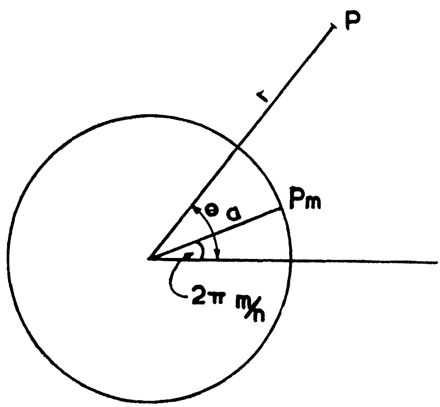


Fig. 46

The currents are through the points

$$P_m, \ (r = a,\ \theta = 2\pi m/n)\ (m = 0, 1, \ldots n - 1)$$

The distance from $P_m$ to the field point $P$ $(r, \theta, 0)$ is given by

$$R_m{}^2 = r^2 + a^2 - 2ar \cos (\theta - 2\pi m/n)$$

and the corresponding vector potential at $P$ is (from 5.1.2.1)

$$-2j\mathbf{k} \log R_m$$

Thus the total vector potential at $P$ is $-j\mathbf{k}\ \Sigma \log R^2{}_m$

$$= -j\mathbf{k} \log \prod_{m=0}^{n-1} \left\{r^2 + a^2 - 2ar \cos\left(\theta - \frac{2\pi m}{n}\right)\right\} \quad . \quad (1)$$

The product is a known expression: most easily arrived at by using

$$Z_m = ae^{2\pi m i/n},\ Z = re^{i\theta}$$

and considering

$$|Z - Z_m|^2 = r^2 + a^2 - 2ar \cos(\theta - 2\pi m/n) = (Z - Z_m)(\bar{Z} - \bar{Z}_m)$$

Now $Z_m$, $\bar{Z}_m$ are roots of $Z^n = a^n$.

Hence $\Pi(Z - Z_m) = Z^n - a^n = r^n e^{in\theta} - a^n$ and $\Pi(\bar{Z} - \bar{Z}_m) = r^n e^{-in\theta} - a^n$ and so $|\Pi(Z - Z_m)|^2 = r^{2n} + a^{2n} - 2r^n a^n \cos n\theta$, the required product (2)

Hence (1) gives

total vector potential $\mathbf{A} = -j\mathbf{k} \log (r^{2n} + a^{2n} - 2r^n a^n \cos n\theta)$ as required.

The force per unit length on the $m^{th}$ wire, at $P_m$ ($r = a$, $\theta = 2\pi m/n$), is

$$j\mathbf{k} \times \mathbf{B}$$

where $\mathbf{B} = \text{curl } \mathbf{A}^1$ and $\mathbf{A}^1$ is the vector potential at $P_m$ due to the other $(n - 1)$ currents.

Thus $\mathbf{A}^1 = -j\mathbf{k} \log \dfrac{r^{2n} + a^{2n} - 2r^n a^n \cos n\theta}{(r - a)^2}$ as $r \longrightarrow a$, $\theta \longrightarrow 2\pi m/n$

and $\mathbf{B}$ has components in the $r$, $\theta$ directions $\dfrac{1}{r}\dfrac{\partial A^1{}_z}{\partial \theta}$, $-\dfrac{\partial A^1{}_z}{\partial r}$

i.e. $$-\frac{j}{r} \cdot \frac{2nr^n a^n \sin n\theta}{(r^{2n} + a^{2n} - 2r^n a^n \cos n\theta)};$$

$$+ j\left\{\frac{2nr^{2n-1} - 2nr^{n-1}a^n \cos n\theta}{r^{2n} + a^{2n} - 2r^n a^n \cos n\theta} - \frac{2(r - a)}{(r - a)^2}\right\}$$

which, putting $\theta = 2\pi m/n$, become

$$0,\ j\left\{\frac{2nr^{2n-1} - 2nr^{n-1}a^n}{(r^n - a^n)^2} - \frac{2}{r - a}\right\}$$

i.e. $$0,\ \frac{2j}{r^n - a^n}\{nr^{n-1} - (r^{n-1} + ar^{n-2} + \ldots + a^{n-1})\}$$

i.e. $$0,\ \frac{2j\{(n - 1)r^{n-2} + (n - 2)ar^{n-3} + \ldots + a^{n-2}\}}{(r^{n-1} + ar^{n-2} + \ldots + a^{n-1})}$$

dividing top and bottom by $(r - a)$. Putting $r = a$ gives

$$\mathbf{B} = 0,\ \frac{2j}{a}\frac{\frac{1}{2}(n - 1)n}{n} = 0,\ j\left(\frac{n - 1}{a}\right)$$

thus the force per unit length which, from 5.1.2.2. (12) is $j\mathbf{k} \times \mathbf{B}$, is $-j^2\left(\frac{n-1}{a}\right)$, i e. is $j^2\left(\frac{n-1}{a}\right)$ towards the $z$-axis.

---

**Problem 65**

Explain what is meant by a plane-polarised electromagnetic wave. Such a wave *in vacuo* is incident at a plane boundary with an isotropic dielectric medium of specific inductive capacity $K$ and permeability $\mu = 1$.

Show that the angles of incidence and reflection are equal and that $\sin \theta = K^{1/2} \sin \phi$, where $\theta$ is the angle of incidence and $\phi$ is the angle of refraction.

If the wave is polarised in the plane of incidence, prove that the ratio of both the electric and magnetic vectors in the reflected and incident rays is $\sin(\phi - \theta)/\sin(\phi + \theta)$. (D.)

---

**Solution.** The speed of propagation in a medium of specific inductive capacity $K$ and permeability is $c/\sqrt{\mu K}$. *In vacuo* this is simply $c$.

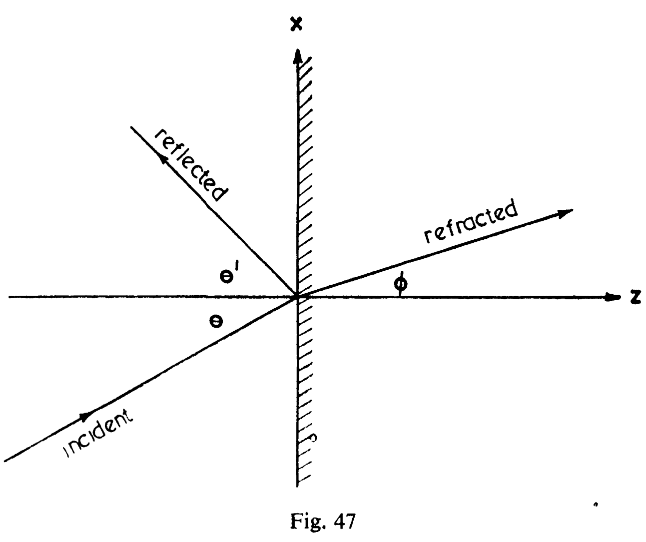


Fig. 47

We consider a wave incident on the plane $z = 0$, $\theta$ being the angle of incidence and $y = 0$ (the plane of the paper) the plane of incidence.

Let $(l_1, m_1, n_1)$, $(l_2, m_2, n_2)$ be the direction cosines of the reflected and refracted rays respectively.

Then, (by 5.7.1), the components of the electric and magnetic

vectors **E** and **H** for the incident, reflected, and refracted rays are proportional to

$$e^{ip\{ct-(x\sin\theta+z\cos\theta)\}},\ e^{ip\{ct-(l_1x+m_1y+n_1z)\}}$$

and $$e^{ip\{ct-\sqrt{K}(l_2x+m_2y+n_2z)\}} \text{ respectively.}$$

We may write the $E_x$ components for instance as

$$ae^{ip\{ct-(x\sin\theta+z\cos\theta)\}},\quad a_1e^{ip\{ct-(l_1x+m_1y+n_1z)\}},$$
$$a_2e^{ip\{ct-\sqrt{K}(l_2x+m_2y+n_2z)\}}$$

But, (5.6.1 (28)), $E_x$ must be continuous in the plane $z = 0$, and so, for all values of $x$, $y$, $t$, we have

$$ae^{ip(ct-x\sin\theta)}+a_1e^{ip\{ct-(l_1x+m_1y)\}}=a_2e^{ip\{ct-\sqrt{K}(l_2x+m_2y)\}}$$

For this the three indices must be identical, and hence

$$\sin\theta = l_1 = \sqrt{K}\,.\,l_2$$

and $$0 = m_1 = \sqrt{K}m_2$$

The second of these two sets of equations shows that the reflected and refracted rays lie in the same plane ($y = 0$) as the incident ray. The first set gives the results:

(i) $\sin\theta = \sin\theta^1$, and hence $\theta = \theta^1$, $\theta^1$ being the angle of reflection, and

(ii) $\sin\theta = \sqrt{K}\sin\phi$, $\phi$ being the angle of refraction, as required.

To solve the last part of the question we must consider the amplitudes of the waves.

It is easier to write down the components of the vectors if we take a new set of axes with $Oz'$ in the direction of the incident ray, $Oy'$

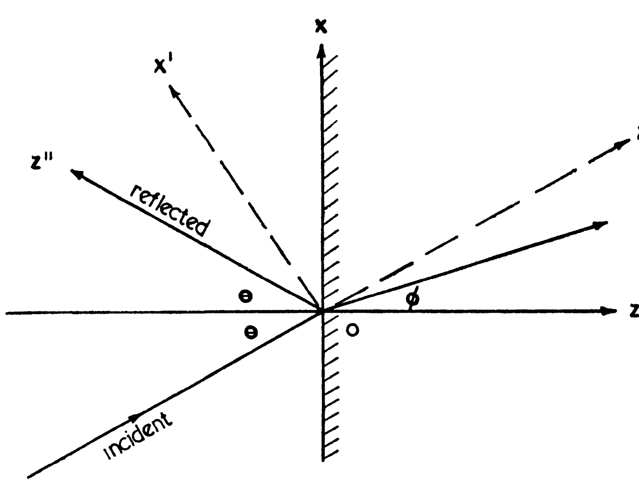


Fig. 48

coincident with $Oy$ (i.e. at right angles to and coming out of the plane of the paper), and $Ox'$ to complete the right-handed triad. Since the wave is polarised in the plane of incidence, **H** lies wholly along $Ox'$ and **E** along $Oy'$.

We have

$$H_{z'} = H_{y'} = 0;\ E_{x'} = E_{z'} = 0$$

Also

$$E_{y'} = Ae^{ip\{ct-(x\sin\theta+z\cos\theta)\}}$$

the index being obtained as in the previous part of the question.

Hence, (by 5.7.1 (33)),

$$H_{x'} = -Ae^{ip\{ct-(x\sin\theta+z\cos\theta)\}}$$

Resolving these quantities to find the components of **E** and **H** along the original axes gives

$$E_x = 0$$
$$E_y = Ae^{ip\{ct-(x\sin\theta+z\cos\theta)\}}$$
$$E_z = 0$$
$$H_x = -A\cos\theta\,.\,e^{ip\{ct-(x\sin\theta+z\cos\theta)\}}$$
$$H_y = 0$$
$$H_z = A\sin\theta\,.\,e^{ip\{ct-(x\sin\theta+z\cos\theta)\}}$$

Treating the *reflected wave* in a similar fashion, choosing new axes $Oz''$ along the direction of reflection, etc., we find that the components of **E** and **H** will be proportional to $e^{ip\{ct-(x\sin\theta-z\cos\theta)\}}$, since the direction cosines of $Oz''$ are $(\sin\theta, 0, -\cos\theta)$.

Carrying through the analysis as for incident wave leads to the results

$$E_x = 0,\ E = Be^{ip\{ct-(x\sin\theta-z\cos\theta)\}},\ E_z = 0$$

and

$$H_x = B\cos\theta\,.\,e^{ip\{ct-(x\sin\theta-z\cos\theta)\}},\ H_y = 0,$$
$$H_z = B\sin\theta\,.\,e^{ip\{ct-(x\sin\theta-z\cos\theta)\}}$$

Similarly, we find for the refracted wave

$$E_x = 0,\ E_y = Ce^{ip\{ct-\sqrt{K}\,.\,(x\sin\phi+z\cos\phi)\}},\ E_z = 0$$

and

$$H_x = -\sqrt{K}C\cos\phi\ e^{ip\{ct-\sqrt{K}\,.\,(x\sin\phi+z\cos\phi)\}}$$
$$H_y = 0$$
$$H_z = \sqrt{K}C\sin\phi\ e^{ip\{ct-\sqrt{K}(x\sin\phi+z\cos\phi)\}}$$

The boundary conditions (5.6.1 (28)) here are that $E_x$, $E_y$, $KE_z$, $H_x$, $H_y$, $H_z$ are all continuous at $z = 0$.

Those for $E_y$ and $H_x$ lead to

$$A + B = C$$

and $$-A\cos\theta + B\cos\theta = -\sqrt{K}C\cos\phi$$

whence $$\frac{A}{\cos\theta + \sqrt{K}\cos\phi} = \frac{B}{\cos\theta - \sqrt{K}\cos\phi}$$

but $\sqrt{K} = \sin\theta/\sin\phi$ and so

$$\frac{A}{B} = \frac{\cos\theta\sin\phi + \sin\theta\cos\phi}{\cos\theta\sin\phi - \sin\theta\cos\phi} \text{ as required.}$$

---

**Problem 66**

The magnetic and electric vectors of a train of plane electromagnetic waves in a medium of dielectric constant $K_1$ and permeability $\mu_1$ are given by

$$\mathbf{H} = (\alpha\mathbf{i} + \beta\mathbf{j})\, e^{ip_0\lambda}, \quad \mathbf{E} = \gamma\mathbf{k}e^{ip_0\lambda},$$

$$\lambda = x\cos\theta + y\sin\theta - c_1 t$$

where $\alpha$, $\beta$, $\gamma$, $p_0$, $\theta$, and $c_1$ are constants. Show that

$$\frac{\alpha}{\sin\theta} = \frac{\beta}{-\cos\theta} = \gamma\sqrt{\frac{K_1}{\mu_1}}$$

A homogeneous medium whose constants are $K_1$, $\mu_1$ occupies the space $x < 0$, the space $x > 0$ being occupied by a similar medium of constants $K_2$, $\mu_2$. If the above wave train is incident upon the interface $x = 0$ of the media and gives rise to similar reflected and transmitted waves propagated with speeds $c_1$, $c_2$ respectively, the angles of reflection and refraction corresponding to $\theta$ being $\psi$ and $\phi$, prove that $\psi = \pi - \theta$, that

$$\frac{c_1}{\sin\theta} = \frac{c_2}{\sin\phi}$$

and that the amplitudes of the incident, reflected and transmitted waves are in the ratios

$$K_1\sin 2\theta + K_2\sin 2\phi : K_1\sin 2\theta - K_2\sin 2\phi : 2\,K_1\sin 2\theta \quad \text{(L.)}$$

---

**Solution.** Since $\lambda = x\cos\theta + y\sin\theta - c_1 t$ we have a plane wave whose direction of propagation is parallel to the $xy$ plane and at an angle $\theta$ with $Ox$.

We know that (5.7.2 (33))

$$\mathbf{n} \times \mathbf{E} = \sqrt{\frac{\mu_1}{K_1}}\,\mathbf{H}$$

and here $$\mathbf{n} = (\cos\theta, \sin\theta, 0)$$

It is given that $$\mathbf{H} = (\alpha e^{ip_0\lambda}, \beta e^{ip_0\lambda}, 0)$$

$$\mathbf{E} = (0, 0, \gamma e^{ip_0\lambda})$$

whence it is easy to write down the equations

$$\gamma \sin \theta = \alpha\sqrt{\mu_1/K_1}$$
$$-\gamma \cos \theta = \beta\sqrt{\mu_1/K_1}$$

providing the results required.

In the second part the angles $\theta$, $\phi$, $\psi$ are measured from the positive $x$-direction and the situation may be represented by this diagram.

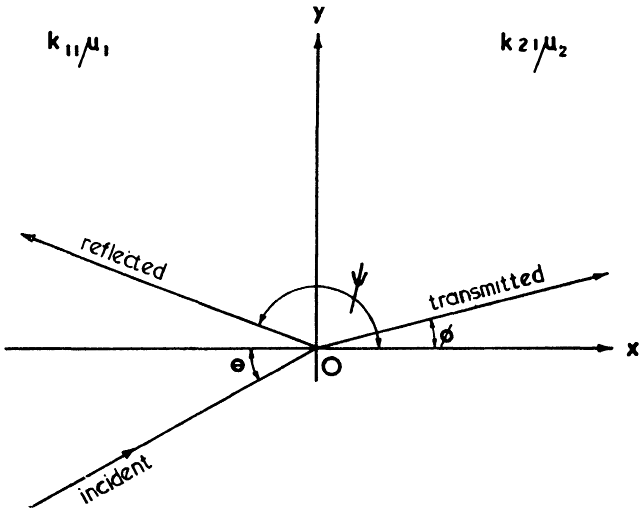


Fig. 49

We can now write down the components of the magnetic and electric vectors as follows (cf. Problem 64):

*Incident wave*

$$H_x = \alpha e^{ip_0\,(x \cos\theta + y \sin\theta - c_1 t)}$$
$$H_y = \beta e^{ip_0\,(x \cos\theta + y \sin\theta - c_1 t)}$$
$$H_z = 0$$
$$E_x = 0 = E_y$$
$$E_z = \gamma e^{ip_0\,(x \cos\theta + y \sin\theta - c_1 t)}$$

Also
$$\frac{\alpha}{\sin\theta} = \frac{\beta}{-\cos\theta} = \gamma\sqrt{\frac{K_1}{\mu_1}} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

*Reflected wave*

$$H_x = \alpha_1 e^{ip_0\,(x \cos\psi + y \sin\psi - c_1 t)}$$
$$H_y = \beta_1 e^{ip_0\,(x \cos\psi + y \sin\psi - c_1 t)}$$
$$H_z = 0$$
$$E_x = E_y = 0$$
$$E_z = \gamma_1 e^{ip_0\,(x \cos\psi + y \sin\psi - c_1 t)}$$

where
$$\frac{\alpha_1}{\sin\psi} = \frac{\beta_1}{-\cos\psi} = \gamma_1\sqrt{\frac{K_1}{\mu_1}} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

*Transmitted wave*

$$H_x = \alpha_2 e^{ip_0 (x\cos\phi + y\sin\phi - c_2 t)}$$
$$H_y = \beta_2 e^{ip_0 (x\cos\phi + y\sin\phi - c_2 t)}$$
$$H_z = 0$$
$$E_x = E_y = 0$$
$$E_z = \gamma_2 e^{ip_0 (x\cos\phi + y\sin\phi - c_2 t)}$$

where $$\frac{\alpha_2}{\sin\phi} = \frac{\beta_2}{-\cos\phi} = \gamma_2\sqrt{\frac{K_2}{\mu_2}} \quad . \quad . \quad . \quad (3)$$

We have to satisfy the boundary conditions that the tangential components of both the magnetic and electric intensities are continuous and that the normal components of the electric displacement, $K\mathbf{E}$, and magnetic induction, $\mu\mathbf{H}$, are continuous.

This implies that at $x = 0$ the indices of the exponential factors must be the same for all $y$, $z$, and $t$.

Hence $$\frac{c_1}{\sin\theta} = \frac{c_1}{\sin\psi} = \frac{c_2}{\sin\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

whence $\psi = \pi - \theta$

The boundary conditions also lead to the following relations between the amplitudes:

$$\left.\begin{array}{ll} E_z & \gamma + \gamma_1 = \gamma_2 \\ H_y & \beta + \beta_1 = \beta_2 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The second of equations (5) becomes, on using (1), (2), and (3),

$$\gamma\sqrt{\frac{K_1}{\mu_1}}\cos\theta - \gamma_1\sqrt{\frac{K_1}{\mu_1}}\cos\theta = \gamma_2\sqrt{\frac{K_2}{\mu_2}}\cos\phi$$

Eliminating $\mu_1$, $\mu_2$ by use of the facts that

$$c_1 = c/\sqrt{\mu_1 K_1},\ c_2 = c/\sqrt{\mu_2 K_2}\ (5.7.1\ (31))$$

gives further

$$c_1 K_1 \gamma \cos\theta - c_1 K_1 \gamma_1 \cos\theta = c_2 K_2 \gamma_2 \cos\phi \quad . \quad . \quad . \quad (6)$$

Solving (5) and (6) for the ratios $\gamma : \gamma_1 : \gamma_2$ and using (4) leads to the desired result.

---

**Problem 67**

Show that **E**, **H** defined by

$$E_x = A_e^{-i\omega(t-uz)},\ E_y = E_z = 0,$$
$$H_x = 0,\ H_y = K^{1/2} A_e^{-i\omega(t-uz)},\ H_z = 0$$

where $A$, $\omega$ are constants, satisfy Maxwell's equations for a uniform medium of dielectric constant $K$ and permeability unity provided that $u = K^{1/2}/c$.

The region $0 \leqslant z \leqslant h$ is filled with material of dielectric constant 4 and permeability 1, and the rest of space is empty. A plane-polarised plane wave of frequency $\omega/2\pi$, travelling in the positive direction of $Oz$, is normally incident on the face $z = 0$. Show that the ratio of the amplitude of this incident wave to the amplitude of the reflected wave in $z < 0$ is

$$(1 + \tfrac{16}{9} \operatorname{cosec}^2 \theta)^{1/2}$$

where $$\theta = 2\omega h/c. \qquad \text{(L.)}$$

---

**Solution.** The first part is essentially bookwork, see (5.7.1, 5.7.2).

For the second part we consider first the region $z < 0$.

There is an *incident wave* for which we may, by the first part, write down the components of **E** and **H**:

$$E_x = A_1 e^{-i\omega(t-z/c)}, \; E_y = 0 = E_z$$

Then, by (5.7.2 (33)),

$$H_x = 0, \; H_y = A_1 e^{-i\omega(t-z/c)}, \; H_z = 0$$

In the same region there is a *reflected wave* which is propagated in the negative direction of $Oz$ (this may be proved as in Problem 65, but is here assumed as a known result). Its components are

$$E_x = A_2 e^{-i\omega(t+z/c)}, \; E_y = 0 = E_z$$

$$H_x = 0, \; H_y = -A_2 e^{-i\omega(t+z/c)}, \; H_z = 0$$

We may treat the regions $0 \leqslant z \leqslant h$ and $h < z$ similarly and obtain:

$\underline{0 \leqslant z \leqslant h.}$

*Transmitted Wave* $\quad E_x = A_3 e^{-i\omega(t-2z/c)}$

$$H_x = 2A_3 e^{-i\omega(t-2z/c)}$$

remembering that $K = 2$, $\mu = 1$ and omitting zero components.

*Reflected Wave* $\quad E_x = A_4 e^{-i\omega(t+2z/c)}$

$$H_x = -2A_4 e^{-i\omega(t+2z/c)}$$

$\underline{h < z}$

*Reflected Wave* $\quad E_x = A_5 e^{-i\omega(t-z/c)}$

$$H_x = A_5 e^{-i\omega(t-z/c)}$$

*Boundary Conditions.* At $z = 0$ and $z = a$ $E_x$ and $H_x$ are both continuous. In the region $z < 0$ the total $E_x$ is the sum of the components of the incident and reflected waves, in $0 \leqslant z \leqslant h$ the sum of the transmitted and reflected waves. Putting $z = 0$, we obtain

$$A_1 + A_2 = A_3 + A_4 \quad . \; . \; . \; . \; . \; . \quad (1)$$

and

$$A_1 - A_2 = 2A_3 - 2A_4 \quad . \; . \; . \; . \; . \; . \quad (2)$$

Proceeding similarly at $z = h$ gives

$$A_3 e^{i\theta} + A_4 e^{-i\theta} = A_5 e^{i\theta/2} \qquad (3)$$
$$2A_3 e^{i\theta} - 2A_4 e^{i\theta} = A_5 e^{i\theta/2} \qquad (4)$$

where $\theta = 2\omega h/c$.

Solving equations (1)–(4) for the ratio $A_1/A_2$ gives

$$A_1/A_2 = (9e^{-i\theta} - e^{i\theta})/3e^{i\theta} - 3e^{i\theta}$$

which reduces to $-5/3 - i\, 4/3 \cot\theta$

Hence the ratio of the amplitudes required,

$$|A_1/A_2| = (\tfrac{25}{9} + \tfrac{16}{9}\cot^2\theta)^{1/2}$$
$$= (1 + \tfrac{16}{9}\operatorname{cosec}^2\theta)^{1/2}$$

---

**Problem 68**

Show that **E**, **H** defined by

$$E_x = 0,\ E_y = a \exp\left\{in\left(\frac{qx}{c} - t\right)\right\},\ E_z = 0$$

$$H_x = 0,\ H_y = 0,\ H_z = qa \exp\left\{in\left(\frac{qx}{c} - t\right)\right\}$$

can represent the electric and magnetic vectors of an electromagnetic wave in a uniform dielectric for which the dielectric constant $K$ is $q^2$ and the permeability $\mu$ is unity.

A dielectric slab of thickness $h$ with plane faces lies in otherwise empty space. A plane-polarised harmonic wave whose electric vector has amplitude $a$ is normally incident on to one face and $b$ is the corresponding amplitude of the emergent transmitted wave. Prove that

$$\frac{a^2}{b^2} = \cos^2\theta + \frac{1}{4}\left(q + \frac{1}{q}\right)^2 \sin^2\theta,$$

where $q^2$ is the dielectric constant of the slab, the permeability of the slab is unity, and $\theta = nqh/c$. (L.)

---

**Solution.** For the first part see 5.7.2.

We set the second part out as in Problem 67.

$\underline{x < 0}$

*Incident Wave* (zero components omitted)

$$E_y = a_1 \exp\{in(x/c - t)\}$$
$$H_z = a_1 \exp\{in(x/c - t)\}$$

*Reflected Wave*

$$E_y = a_2 \exp\{-in(x/c + t)\}$$
$$H_z = -a_2 \exp\{-in(x/c + t)\}$$

$0 \leqslant x \leqslant h$

*Transmitted Wave*

$$E_y = a_3 \exp\{in(qx/c - t)\}$$
$$H_z = qa_3 \exp\{in(qx/c - t)\}$$

*Reflected Wave*

$$E_y = a_4 \exp\{-in(qx/c + t)\}$$
$$H_z = -qa_4 \exp\{-in(qx/c + t)\}$$

$h < x$

*Transmitted Wave*

$$E_y = a_5 \exp\{in(x/c - t)\}$$
$$H_z = a_5 \exp\{in(x/c - t)\}$$

Applying the usual boundary conditions (as in the previous problem) at $x = 0$, $x = h$ gives

$$\begin{aligned} a_1 + a_2 &= a_3 + a_4 \\ a_1 - a_2 &= qa_3 - qa_4 \\ a_3e^{i\theta} + a_4e^{-i\theta} &= a_5e^{i\theta/q} \\ qa_3e^{i\theta} - qa_3e^{-i\theta} &= a_5e^{i\theta/q} \end{aligned}$$

where $\theta$ has been written for $nqh/c$. Solving these for the ratio $a_1/a_5e^{i\theta/q}$ leads to

$$\frac{a_1}{a_5e^{i\theta/q}} = \cos\theta - i\frac{q^2+1}{2q}\sin\theta$$

whence $$\left|\frac{a_1}{a_5e^{i\theta/q}}\right|^2 = \frac{a^2}{b^2} = \cos^2\theta + \frac{1}{4}\left(q + \frac{1}{q}\right)^2 \sin^2\theta$$

---

**Problem 69**

Prove that in a medium of conductivity $\sigma$, unit permeability and unit dielectric constant, there is a solution of Maxwell's equations of the form

$$\mathbf{E} = \mathbf{i}E\exp(i\omega t - in\omega z/c - \alpha\omega z/c)$$
$$\mathbf{H} = \mathbf{j}H\exp(i\omega t - in\omega z/c - \alpha\omega z/c)$$

where $H = (n - i\alpha)E$ and $(n - i\alpha)^2 = 1 - 4\pi i\sigma/\omega$.

Material of conductivity $\sigma$ fills the region $0 \leqslant z$, and a plane harmonic wave of frequency $\omega/2\pi$, given by

$$\mathbf{E} = \mathbf{i}A\exp(i\omega t - i\omega z/c)$$
$$\mathbf{H} = \mathbf{j}A\exp(i\omega t - i\omega z/c)$$

in the region $z \leqslant 0$, is incident on the face $z = 0$. Prove that the ratio of the energy of the wave reflected by the material to the energy of the incident wave is $(n - 1)/(n + 1)$. Find the fraction of the incident energy which crosses a plane distant $z(> 0)$ from the surface of separation. (L.)

---

**Solution.** The first part is easily obtainable from 5.7.3 (36) and the analysis of Problem 64.

The analysis for the second part is as in Problem 64, with suitable modifications.

$z < 0$

*Incident Wave*

$$E_x = A \exp i\omega(t - z/c)$$
$$H_y = A \exp i\omega(t - z/c)$$

*Reflected Wave*

$$E_x = A \exp i\omega(t + z/c)$$
$$H_y = -A \exp i\omega(t + z/c)$$

$z > 0$

*Refracted Wave*

$$E_x = A \exp(i\omega t - in\omega z/c - \alpha\omega z/c)$$
$$H_y = (n - i\alpha)A \exp(i\omega t - in\omega z/c - \alpha\omega z/c)$$

By continuity of $E_x$ and $H_y$ at $z = 0$ we obtain

$$A_1 + A_2 = A_3 \quad \ldots \ldots \quad (1)$$
$$A_1 - A_2 = (n - i\alpha)A_3 \quad \ldots \ldots \quad (2)$$

Using equ. (34) of 5.7.2 for the energy and remembering that $K = \mu = 1$ *in vacuo*, we find from (1) and (2) that

$$\frac{\text{Reflected energy}}{\text{Incident energy}} = \left|\frac{A_2}{A_1}\right|^2 = \frac{(n-1)^2 + \alpha^2}{(n+1)^2 + \alpha^2},$$

But $(n - i\alpha)^2 = 1 - 4\pi i\sigma/\omega$, whence $n^2 - \alpha^2 = 1$.

Hence $$\left|\frac{A_2}{A_1}\right|^2 = \frac{2n^2 - 2n}{2n^2 + 2n} = \frac{n-1}{n+1}$$

In the conducting medium there is a decay factor $e^{-\alpha\omega z/c}$ in both the electric and magnetic intensities.

It follows that since the fraction of incident energy which is transmitted is $2/(n + 1)$, the fraction crossing the plane $z(> 0)$ is $2e^{-2\alpha\omega z/c}/(n + 1)$.

---

**Comment.** The mean flux of energy in the *refracted* wave may be obtained directly by use of the complex Poynting vector (see, e.g., Ferraro, p. 517) as the real part of $\frac{c}{8\pi}\,\mathbf{E}_0 \times \overline{\mathbf{H}}_0$.

This is the form applicable when $\mathbf{E}$, $\mathbf{H}$ are products of $\mathbf{E}_0$, $\mathbf{H}_0$ and a complex exponential, and $\mathbf{E}$ and $\mathbf{H}$ are complex vectors. The ratio of the flux of incident to reflected energy is, however, simply the ratio of the squares of the moduli, as given, since in this case

$$\begin{aligned} &\text{Incident:} && \mathbf{E}_0 \times \overline{\mathbf{H}}_0 = A_1\bar{A}_1 = |A_1|^2 \\ &\text{Reflected:} && \mathbf{E}_0 \times \overline{\mathbf{H}}_0 = A_2\bar{A}_2 = |A_2|^2 \end{aligned}$$

---

**Problem 70**

Verify that Maxwell's equations for free space are satisfied by

$$\mathbf{H} = \frac{1}{c}\frac{\partial}{\partial t}\,\text{grad}\,\phi \times \mathbf{k}$$

$$\mathbf{E} = -\mathbf{k}\,\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{\partial}{\partial z}\,\text{grad}\,\phi$$

where $\mathbf{k}$ is the unit vector along the $z$-axis and $\phi$ satisfies $\nabla^2\phi = \dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial t^2}$

Show that $\phi$ can be of the form

$$A \sin \alpha x \sin \beta y \cos \gamma z \cos \omega t$$

provided that the angular frequency $\omega$ and the constants $\alpha$, $\beta$, $\gamma$ satisfy a certain relation. Find the Cartesian components of $\mathbf{E}$ in this case and deduce that such a field can exist in the region $0 \leqslant x \leqslant l$, $0 \leqslant y \leqslant l$, $0 \leqslant z \leqslant l$, where the boundaries are perfectly conducting, if $\alpha$, $\beta$, $\gamma$ have suitably chosen values. If the field does not vanish identically, show that the least allowed value of $\omega$ is $(\pi c/l)\sqrt{2}$. (L.)

---

**Solution.** In free space Maxwell's equations are

$$\text{div}\,\mathbf{H} = \text{div}\,\mathbf{E} = 0;\ \text{curl}\,\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

$$\text{curl}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$$

With the given value of $\mathbf{H}$

$$\begin{aligned} \text{div}\,\mathbf{H} &= \frac{1}{c}\frac{\partial}{\partial t}\,\text{div}\,(\text{grad}\,\phi \times \mathbf{k}) \\ &= \frac{1}{c}\frac{\partial}{\partial t}(\mathbf{k}\,.\,\text{curl grad}\,\phi - \text{grad}\,\phi\,.\,\text{curl}\,\mathbf{k})\ \text{(by 1.2)} \end{aligned}$$

But curl $\mathbf{k} = 0$ ($\mathbf{k}$ constant) and curl grad $\phi \equiv 0$ (1.2).

$$\therefore \qquad \text{div } \mathbf{H} = 0$$

Similarly, div $\mathbf{E} = -\frac{1}{c^2}\text{div}\left(\frac{\partial^2\phi}{\partial t^2}\mathbf{k}\right) + \frac{\partial}{\partial z}(\nabla^2\phi)$

(as div grad $\phi \equiv \nabla^2\phi$)

$$= -\frac{1}{c^2}\frac{\partial}{\partial z}\left(\frac{\partial^2\phi}{\partial t^2}\right) + \frac{\partial}{\partial z}\left(\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}\right)$$
$$= 0$$

curl $\mathbf{E} = -\frac{1}{c^2}\left(\mathbf{i}\frac{\partial}{\partial y} - \mathbf{j}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2\phi}{\partial t^2}\right)$ (as curl grad $\equiv 0$ and using 1.1.5)

$$-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left\{\left(\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}\right) \times \mathbf{k}\right\}$$
$$= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(-\mathbf{j}\frac{\partial\phi}{\partial x} + \mathbf{i}\frac{\partial\phi}{\partial y}\right)$$

= curl $\mathbf{E}$ as required.

(Since the order of differentiation in space and time may be reversed.)

Finally, curl $\mathbf{H} = \frac{1}{c}\frac{\partial}{\partial t}\{\text{curl (grad } \phi \times \mathbf{k})\}$

$$= \frac{1}{c}\frac{\partial}{\partial t}\{(\mathbf{k}\,.\,\nabla)\text{ grad }\phi$$
$$-(\text{grad }\phi\,.\,\nabla)\,\mathbf{k} + \text{grad }\phi \text{ div }\mathbf{k} - \mathbf{k}\,\nabla^2\phi\} \text{ (by 1.2)}$$
$$= \frac{1}{c}\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial z}\text{grad }\phi - \mathbf{k}\nabla^2\phi\right\}$$
$$= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

It is easy to verify by straightforward differentiation that $\phi$ can be of the form $A \sin \alpha x \sin \beta y \cos \gamma z \cos \omega t$ provided that

$$\omega^2 = c^2(\alpha^2 + \beta^2 + \gamma^2)$$

Substitution for $\phi$ now gives

$$E_x = -A\alpha\gamma \cos \alpha x \sin \beta y \sin \gamma z \cos \omega t \quad . \quad . \quad . \quad . \quad (1)$$
$$E_y = -A\beta\gamma \sin \alpha x \cos \beta y \sin \gamma z \cos \omega t \quad . \quad . \quad . \quad . \quad (2)$$
$$E_z = A(\omega^2/c^2 - \gamma^2) \sin \alpha x \sin \beta y \cos \gamma z \cos \omega t$$
$$= A(\alpha^2 + \beta^2) \sin \alpha x \sin \beta y \cos \gamma z \cos \omega t \quad . \quad . \quad (3)$$

On the boundary planes of the cubical region given we must have (since they are perfect conductors)

$$E_y = E_z = 0 \text{ at } x = 0 \text{ and } x = l$$
$$E_z = E_x = 0 \text{ at } y = 0 \text{ and } y = l$$
$$E_x = E_y = 0 \text{ at } z = 0 \text{ and } z = l$$

These are satisfied if

$$\sin \alpha l = 0 = \sin \beta l = \sin \gamma l$$

i.e.

$$\alpha l = n_1\pi,\ \beta l = n_2\pi,\ \gamma l = n_3\pi$$

where $n_1$, $n_2$, $n_3$ are integers, no two of which may be zero if the field is not to vanish identically. Hence the least value for $\omega$ is given by

$$\omega^2 = c^2(\pi^2/l^2 + \pi^2/l^2)$$

i.e.

$$\omega = \frac{\pi c\sqrt{2}}{l}$$

---

**Problem 71**

Verify that a possible form of the electric intensity vector **E** in the free space between two perfectly conducting planes $z = 0$ and $z = a$ is

$$E_x = E \cos \alpha \sin (kz \cos \alpha) \sin \{k(x \sin \alpha - ct)\}$$
$$E_y = 0$$
$$E_z = E \sin \alpha \cos (kz \cos \alpha) \cos \{k(x \sin \alpha - ct)\}$$

where $E$ and $\alpha$ ($\neq \frac{1}{2}\pi$) are constants. Show that $k$ must satisfy a certain equation. Find the corresponding magnetic intensity vector **H** containing only periodic terms. Prove that the energy flux is entirely in the $x$ direction and that the average flux across the area bounded by $y = 0$, $y = b$, $z = 0$, and $z = a$ is $(abcE^2 \sin \alpha)/16\pi$. (L.)

---

**Solution.** Maxwell's equations in free space are

$$\text{div } \mathbf{E} = 0;\ \text{div } \mathbf{H} = 0,$$

$$\text{curl } \mathbf{H} = \frac{1}{c}\frac{\partial E}{\partial t};\ \text{curl } \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$$

and these may be combined to give $\nabla^2\mathbf{E} = \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{E}}{\partial t^2}$ (5.7.1 (30)).

The given value of **E** may be shown by direct differentiation to satisfy this, and also div **E** = 0. Thus it is a possible form for **E** in free space (though we have still to show that it satisfies the boundary conditions). To find **H** we have

$$\text{curl } \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = -\mathbf{i}Ek \cos \alpha \sin(kz \cos \alpha) \cos k(x \sin \alpha - ct)$$
$$+\mathbf{k}Ek \sin \alpha \cos(kz \cos \alpha) \sin k(x \sin \alpha - ct) \quad (1)$$

$$-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = \text{curl } \mathbf{E} = \mathbf{i}(0) + \mathbf{j}\{Ek \cos^2 \alpha \cos (kz \cos \alpha) \sin k(x \sin \alpha - ct)$$
$$+Ek \sin^2 \alpha \cos(kz \cos \alpha) \sin k(x \sin \alpha - ct)\} + \mathbf{k}(0) \quad . \quad (2)$$

From (2) we find

$$\frac{\partial H_x}{\partial t} = 0 = \frac{\partial H_z}{\partial t}$$

and $\dfrac{\partial H_y}{\partial t} = -Ekc \cos(kz \cos \alpha) \sin k(x \sin \alpha - ct)$

$$\left.\begin{array}{ll} \text{whence we have} & H_x = H_z = 0 \\ \text{and} & H_y = -E \cos(kz \cos \alpha) \cos k(x \sin \alpha - ct) \end{array}\right\} \quad . \quad . \quad (3)$$

It is readily verified that this satisfies (1) and also that div $\mathbf{H} = 0$.

On $z = 0$ and $z = a$ we must have $E_x = 0$.

This is obvious at $z = 0$, but at $z = a$ we require $\sin(ka \cos \alpha) = 0$ (as $\alpha \neq \pi/2$) or $ka \cos \alpha = n\pi$ ($n$ an integer).

The energy flux is measured by the time average of Poynting's vector $\dfrac{c}{4\pi}(\mathbf{E} \times \mathbf{H})$, which in this case reduces to

$$\frac{c}{4\pi}(-\mathbf{i}E_zH_y + \mathbf{k}E_xH_y)$$

The time average of the first component integrated over the area bounded by $z = 0$, $a$, and $y = 0$, $b$ is

$$\frac{kc}{2\pi} \cdot \frac{cE^2 \sin \alpha}{4\pi} \int_0^b dy \int_0^a \tfrac{1}{2}\{1 + \cos(2kz \cos \alpha)\}dz \int_0^{2\pi/kc} \tfrac{1}{2}\{1 + \cos 2k(x \sin \alpha - ct)\}dt$$

and, remembering that $ka \cos \alpha = n\pi$, this reduces to $abcE^2 \sin \alpha/16\pi$, as required.

Treating the other component similarly gives zero flow in the $z$-direction.

---

**Problem 72**

Deduce from Maxwell's equations that $v$, the velocity of propagation of electromagnetic waves in an infinite uniform medium of dielectric constant $K$ and permeability $\mu$, is equal to $c/(\mu K)^{1/2}$, where $c$ is the ratio of the e.m.u. to the e.s.u. of electric charge.

A long circular cylinder of this material, of radius $a$, is embedded in a perfect conductor, the axis of the cylinder being the $z$-axis. Assuming that in the dielectric $H_z = 0$, and all other field quantities are of the form $R(r)\, \Theta(\theta) \exp(i\omega t - \gamma z)$, where $\omega$, $\gamma$ are constants and $(r, \theta, z)$ are cylindrical polar coordinates, show that $J_n(\nu a) = 0$, where $J_n(x)$ is the Bessel function of the first kind, $n$ being an integer, and

$$\nu^2 = \left(\frac{\omega}{v}\right)^2 + \gamma^2$$

Show that, for a given $n$, there is a critical value $\omega_0$ which $\omega$ must exceed if waves of this type are to be transmitted in the cylinder. (D.)

---

**Solution.** The first part of this is a standard result.

For the second part, consider the equations satisfied by **E** and **H**, which are, by Maxwell's equations (5.6),

$$\operatorname{curl}\mathbf{H} = \frac{K}{c}\frac{\partial\mathbf{E}}{\partial t},\ \operatorname{curl}\mathbf{E} = -\frac{\mu}{c}\frac{\partial\mathbf{H}}{\partial t}\ \text{which reduce to}\ \frac{1}{v^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \nabla^2\mathbf{E} \quad (1)$$

where, using cylindrical coordinates $r$, $\theta$, $z$

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2} \quad . \quad . \quad . \quad . \quad (2)$$

and we are to take

$$\mathbf{E} = \mathbf{E}_0 \exp(i\omega t - \gamma z),\ \mathbf{H} = \mathbf{H}_0 \exp(i\omega t - \gamma z) \quad . \quad (3)$$

with $\mathbf{E}_0$, $\mathbf{H}_0$ of the form $R(r)\,\Theta(\theta)$ and $H_{0z} = 0$ . . (4)

Substituting (3) in (1) gives, for $E_{0z}$

$$-\frac{\omega^2}{v^2}E_{0z} = (\nabla^2 + \gamma^2)E_{0z}$$

and substituting (4) and (2) in this, and separating the variables

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -\left(\frac{\omega^2}{v^2} + \gamma^2\right)r^2 - \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -n^2 \quad . \quad (5)$$

where $n$ is a constant by the usual argument.

The solutions must

(i) be periodic in $\theta$ with period a multiple of $2\pi$ — since the intensity is a single valued function of position,

(ii) satisfy $E_{0z} = 0$ on $r = a$, i.e. $R = 0$ on $r = a$, since the tangential component of intensity must vanish at the surface of a conductor.

Thus we must have from (i)

$$\Theta = A\cos n\theta + B\sin n\theta$$

where $n$ is an *integer*.

And, since (5) is

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left\{\left(\frac{\omega^2}{v^2} + \gamma^2\right)r^2 - n^2\right\} = 0$$

which is Bessel's equation, the solution is

$$R = J_n(\nu r)$$

where $\nu^2 = \omega^2/v^2 + \gamma^2$

Applying condition (ii), we must have

$$J_n(\nu a) = 0 \text{ as required} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For given $n$, condition (6) defines a series of possible (positive) values of $\nu a$, (i.e. the positive zeros of the Bessel function $J_n$.) Then

$$\gamma^2 = \nu^2 - \omega^2/v^2$$

and the wave is transmitted without attenuation only if $\gamma$ is purely imaginary, i.e. if $\omega^2 > v^2\nu^2$. If $\nu_0$ be the smallest possible value of $\nu$, corresponding to the smallest positive zero, then $\omega > \omega_0$ where

$$\omega_0 = v\nu_0$$

---

**Problem 73**

Write down Maxwell's equations for free space and show that there is a possible solution given by

$$\mathbf{E} = \text{curl curl } (\mathbf{k}\phi)$$
$$\mathbf{H} = \text{curl } (\dot{\phi}\mathbf{k}/c)$$

where $\mathbf{k}$ is a fixed unit vector and $\phi$ is a scalar function that satisfies the wave equation $\nabla^2 \phi = \ddot{\phi}/c^2$.

An electric dipole at the origin has a moment $\mathbf{k}M(t)$, which varies with the time. Find the function $\phi$ for the field produced by this dipole and show that the equation of a line of electric force in a plane containing the origin and the polar axis is, in polar coordinates,

$$C^{-1}\,\dot{M}(t - r/c) + r^{-1}\,M\left(t - \frac{r}{c}\right) = A \operatorname{cosec} 2\theta$$

where $A$ is a constant, $\theta$ being measured from the direction of $\mathbf{k}$. (L.)

---

**Solution.** Maxwell's equations (5.6) for free space are, if $K = \mu = 1$

$$\text{curl } \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \text{div } \mathbf{B} = 0 \qquad \mathbf{B} = \mathbf{H}$$

$$\text{curl } \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \qquad \text{div } \mathbf{D} = 0 \qquad \mathbf{D} = \mathbf{E}$$

Consider $\mathbf{E} = \text{curl curl } \mathbf{k}\phi = \text{grad div } (\mathbf{k}\phi) - \nabla^2\mathbf{k}\phi$

and $\mathbf{H} = \text{curl } (\dot{\phi}\mathbf{k}/c)$

$\mathbf{k}$ being a fixed unit vector.

Then the vector relations in 1.2 give

(1) $\text{div } \mathbf{E} = \text{div curl (curl } \mathbf{k}\phi) = 0$ identically

(2) $\text{div } \mathbf{H} = \text{div curl } (\dot{\phi}\mathbf{k}/c) = 0$ identically

(3) $\text{curl } \mathbf{H} = \text{curl curl } (\dot{\phi}\mathbf{k}/c) = \text{grad div } (\dot{\phi}\mathbf{k}/c) - \nabla^2(\dot{\phi}\mathbf{k}/c)$

$$= \frac{1}{c}\text{ grad div } (\dot{\phi}\mathbf{k}) - \frac{1}{c}\nabla^2(\mathbf{k}\dot{\phi})$$

$$= \frac{1}{c}\frac{\partial E}{\partial t}$$

since the space and time differential operators are independent.

(4) $\text{curl } \mathbf{E} = \text{curl } \{\text{grad div } (\mathbf{k}\phi) - \nabla^2\mathbf{k}\phi\} = -\text{curl } \nabla^2\mathbf{k}\phi$

$$\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}\text{ curl } (\dot{\phi}\mathbf{k}/c) = \frac{1}{c^2}\text{ curl } (\mathbf{k}\ddot{\phi})$$

Thus $\text{curl } E = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$ if $-\text{ curl } \nabla^2\mathbf{k}\phi = -\frac{1}{c^2}\text{ curl } \mathbf{k}\ddot{\phi}$

and this is certainly true if $\nabla^2\phi = \frac{1}{c^2}\ddot{\phi}$ as required.

An electric dipole at the origin has moment $\mathbf{M} = \mathbf{k}M(t)$. Since this will radiate, we require a solution of the wave equation in terms of the radial distance $r$ and time $t$ which reduces to the known field for a dipole as $r \longrightarrow 0$. We try the standard form of solution

$$\phi = \frac{M(t - r/c)}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Then $$\mathbf{E} = \text{grad div } (\mathbf{k}\phi) - \nabla^2\mathbf{k}\phi$$

and since $\phi = \phi(r, t)$, this gives after some manipulation

$$\mathbf{E} = \left(\frac{\mathbf{k} \cdot \mathbf{r}}{r}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}\right)\mathbf{r} + \frac{1}{r}\frac{\partial\phi}{\partial r}\mathbf{k} - \left(\frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial r^2}\right)\mathbf{k}$$

which gives, on substituting for $\phi$ from (1),

$$\mathbf{E} = \left(\frac{\mathbf{k} \cdot \mathbf{r}}{r}\right)(3M/r^4 + 3M'/cr^3 + M''/c^2r^2)\mathbf{r} - (M/r^3 + M'/cr^2 + M''/c^2r)\mathbf{k} \quad . \quad (2)$$

The leading terms of this as $r \longrightarrow 0$ are

$$\mathbf{E} = 3\frac{(\mathbf{k} \cdot \mathbf{r})M}{r^5}\mathbf{r} - \frac{M\mathbf{k}}{r^3}$$

which give the electric field due to a dipole $M\mathbf{k}$ as required. Hence (1) is the value of $\phi$ for the field produced by a variable dipole.

A line of electric force is a curve whose tangent at any point is in the direction of the vector $\mathbf{E}$ at that point, and, by (2),

$$\mathbf{E} = A_1\mathbf{r} + A_2\mathbf{k}$$

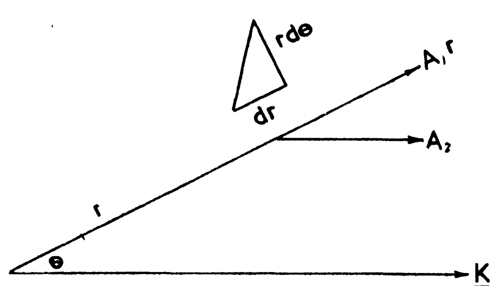


Fig. 50

Thus the condition is

$$\frac{dr}{A_1 r + A_2 \cos\theta} = \frac{r d\theta}{-A_2 \sin\theta}$$

where $A_1 r = \cos\theta\,(3M/r^3 + 3M'/cr^2 + M''/c^2 r)$,

$$A_2 = -M/r^3 - M'/cr^2 - M''/c^2 r$$

so that

$$\frac{dr}{(2M/r^3 + 2M'/cr^2)}\cos\theta = \frac{r d\theta}{(M/r^3 + M'/cr^2 + M''/c^2 r)\sin\theta}$$

which may be put into the form

$$2\cot\theta\, d\theta + \frac{d(M'/c + M/r)}{M'/c + M/r} = 0$$

and so integrates to

$$M'/c + M/r = A\,\mathrm{cosec}^2\,\theta \text{ as required}$$

---

**Comment.** The general expression $\frac{1}{r}f(t - r/c)$ may be shown by direct differentiation to be a solution of the wave equation $\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$.

---

**Problem 74**

Assuming that the Lagrangian function for a particle of mass $m$ and charge $e$ moving in an electromagnetic field is

$$L = \tfrac{1}{2}m\mathbf{v}^2 - e\phi + e(\mathbf{V}\,.\,\mathbf{A})/c$$

where $\phi$ and $\mathbf{A}$ are the scalar and vector potentials of the field, find the force on the particle. Prove that the path of a particle moving in a uniform magnetic field $H_0\mathbf{k}$ is a helix of the form

$$x = (U/\omega)\sin\omega t,\; y = (U/\omega)\cos\omega t,\; z = Wt$$

where $\omega = eH_0/mc$, and $(U, 0, W)$ are the components of the velocity of projection.

If the field is non-uniform, being given by

$$\mathbf{H} = H_0(\varepsilon y\mathbf{i} + \varepsilon x\mathbf{j} + \mathbf{k})$$

where $\varepsilon$ is small, prove that components $(u, v, w)$ of the velocity at time $t$ are, approximately,

$$u = U\cos\omega t,\; v = -U(1 - \varepsilon W/\omega)\sin\omega t,\; w = W$$

the circumstances of projection being the same as above. Find the coordinates of the particle at time $t$ to the first order in $\varepsilon$.

(The unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ denote the directions of the coordinate axes.)

(L.)

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**Solution.** In terms of $x$, $y$, $z$ and their time-derivatives

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - e\phi(x, y, z) + \frac{e}{c}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z)$$

Then the equation of motion in the $x$-direction is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

i.e.

$$\frac{d}{dt}\left(m\dot{x} + \frac{e}{c}A_x\right) = -e\frac{\partial \phi}{\partial x} + \frac{e}{c}\left(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x}\right) \quad . \quad (1)$$

Here $\frac{d}{dt}$ denotes differentiation following the motion of the charge (compare the derivation of the equations of motion in hydrodynamics).

Hence $$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z}$$

Thus (1) becomes

$$m\ddot{x} + \frac{e}{c}\frac{\partial A_x}{\partial t} + \frac{e}{c}\dot{y}\frac{\partial A_x}{\partial y} + \frac{e}{c}\dot{z}\frac{\partial A_x}{\partial z} = -e\frac{\partial \phi}{\partial x} + \frac{e}{c}\dot{y}\frac{\partial A_y}{\partial x} + \frac{e}{c}\dot{z}\frac{\partial A_z}{\partial x}$$

or

$$m\ddot{x} = \left(-\frac{e}{c}\frac{\partial A_x}{\partial t} - e\frac{\partial \phi}{\partial x}\right) + \frac{e}{c}\left\{\dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \dot{z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right\} \quad (2)$$

Combining (2) with two similar equations for $m\ddot{y}$, $m\ddot{z}$ and writing the result in vectors gives

$$m\dot{\mathbf{v}} = -\frac{e}{c}\frac{\partial \mathbf{A}}{\partial t} - e \text{ grad } \phi + \frac{e}{c}\mathbf{V} \times \text{curl } \mathbf{A}$$

or $$m\dot{\mathbf{v}} = e\left(\mathbf{E} + \frac{1}{c}\mathbf{V} \times \mathbf{H}\right) \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

Clearly the total force on the charge is the right-hand side of (3).

When $\mathbf{H} = H_0\mathbf{k}$ ($\mathbf{E} = 0$), (3) gives the equations of motion

$$\ddot{x} = \frac{eH_0}{mc}\dot{y} = \omega\dot{y},\ \ddot{y} = -\omega\dot{x},\ \ddot{z} = 0$$

which, using the given initial conditions $\dot{x} = U$, $\dot{y} = 0$, $\dot{z} = W$, $x = 0$, $y = U/\omega$, $z = 0$, $t = 0$, readily integrate to give the required equations for the path of the particle.

When $\mathbf{H} = H_0(\varepsilon y\mathbf{i} + \varepsilon x\mathbf{j} + \mathbf{k})$ the equations of motion are

$$\begin{aligned} \ddot{x} &= \omega(\dot{y} - \varepsilon x\dot{z}) \\ \ddot{y} &= \omega(-\dot{x} + \varepsilon y\dot{z}) \\ \ddot{z} &= \omega\varepsilon(x\dot{x} - y\dot{y}) \quad . \; . \; . \; . \; . \; . \quad (4) \end{aligned}$$

The third of these integrates to give, with the same initial conditions as above,

$$\dot{z} = \tfrac{1}{2}\omega\varepsilon(x^2 - y^2) + W + \tfrac{1}{2}\frac{\varepsilon U^2}{\omega} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Substituting in the first two equations of (4), rejecting terms in $\varepsilon$ of order higher than the first, gives

$$\left.\begin{aligned} \ddot{x} &= \omega\dot{y} - W\omega\varepsilon x \\ \ddot{y} &= -\omega\dot{x} + W\omega\varepsilon y \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Now assume

$$x = \frac{U}{\omega}\sin\omega t + \xi$$

$$y = \frac{U}{\omega}\cos\omega t + \eta$$

where $\xi$ and $\eta$ will be of order $\varepsilon$.

Substitution in (6) gives, to the same order,

$$\left.\begin{aligned} \ddot{\xi} - \omega\dot{\eta} &= -W\varepsilon U\sin\omega t \\ \ddot{\eta} + \omega\dot{\xi} &= W\varepsilon U\cos\omega t \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

whence, integrating once

$$\left.\begin{aligned} \dot{\xi} - \omega\eta &= \frac{W\varepsilon U}{\omega}\cos\omega t - \frac{W\varepsilon U}{\omega} \\ \dot{\eta} + \omega\xi &= \frac{W\varepsilon U}{\omega}\sin\omega t \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

as $\xi = \eta = 0 = \dot{\xi} = \dot{\eta}$ when $t = 0$.

The second equation of (7) with the first of (8) gives

$$\ddot{\eta} + \omega^2\eta = W\varepsilon U$$

whence

$$\eta = \frac{W\varepsilon U}{\omega^2}(1 - \cos\omega t)$$

using the initial conditions.

Hence
$$y = \frac{U}{\omega}\cos\omega t + \frac{W\varepsilon U}{\omega^2}(1 - \cos\omega t)$$

whence
$$v = \dot{y} = -U\sin\omega t + \frac{W\varepsilon U}{\omega}\sin\omega t$$

It now follows easily that

$$\xi = 0, \text{ whence } x = \frac{U}{\omega} \sin \omega t$$

and $u = \dot{x} = U \cos \omega t$.

Equation (5) now yields

$$\dot{z} = W + \tfrac{1}{2}\varepsilon U^2/\omega - \tfrac{1}{2}\varepsilon U^2 \cos 2\omega t/\omega$$

whence

$$z = Wt + \tfrac{1}{2}\varepsilon U^2 t/\omega + \tfrac{1}{4}\varepsilon U^2 \sin 2\omega t/\omega^2$$

## PROBLEMS FOR SOLUTION

1. Determine the radial and transverse components of force due to a small magnet of moment $m$.

A circular wire of radius $a$ carries a current $i$. Another wire in the form of a small square of side $b$ carries a current $i^1$, and is distant $L$ from the centre of the circle along the normal to the plane of the circle through its centre. The planes of the wires are inclined at an angle $\theta$. Determine the force and couple on the small circuit.

What is the mutual induction between the two circuits? (O.)

2. Show that the magnetic vector potential for a uniform magnetic field $(0, 0, H)$ has components $A_r$, $A_\theta$, $A_\phi$ in spherical polar coordinates given by $A_r = 0$, $A_\theta = 0$, $A_\phi = \tfrac{1}{2}Hr \sin \theta$.

Show also that, for the field due to a small magnet of moment $M$ at the origin pointing along the axis of the coordinates, the vector potential is

$$A_r = 0, A_\theta = 0, A_\phi = Mr^{-2} \sin \theta$$

A sphere is uniformly magnetised with permanent magnetisation of intensity $\mathbf{M}$ and is placed *in vacuo* in a uniform magnetic field of intensity $\mathbf{H}$. Prove that the resultant couple on the sphere is $\tfrac{4}{3}\pi a^3 \mathbf{M} \times \mathbf{H}$. (O.)

3. Obtain the Biot–Savart law from the expression $\mathbf{A} = \dfrac{I}{c}\displaystyle\int \frac{ds}{r}$ for the vector potential of a current loop.

A plane loop, through which a current $I$ e.s.u. flows, consists of a square of side $2a$. Find the magnetic field at a point which is distant $b$ from the plane of the loop and lies on the normal through the centre of the square. (M.)

4. Derive formulae for the electric and magnetic intensities of an electric dipole oscillating with frequency $\omega/2\pi$ in free space. At large distances from the dipole, show that the field is transverse to the direction of propagation and find the Poynting vector. (M.)

5. A plane harmonic, plane-polarised electromagnetic wave of frequency $\omega/2\pi$ in a medium of dielectric constant $\varepsilon_1$ and permeability $\mu_1$, is incident on to the plane face of a semi-infinite medium of dielectric constant $\varepsilon_2$ and permeability $\mu_2$. The direction of propagation of the wave makes an angle $\theta$ with the normal to the interface and the electric field of the wave is normal to the plane of incidence and has unit amplitude.

Find the magnetic field of the incident wave and the magnetic and electric fields of the reflected and transmitted waves. Also obtain an expression for the difference in phase between incident and reflected waves, when total reflection occurs. (M.)

6. Electromagnetic waves of frequency $\omega/2\pi$ are propagated along a rectangular wave-guide, having perfectly conducting walls $x = 0$, $x = a$, $y = 0$, $y = b$, and containing a medium of dielectric constant $K$ and magnetic permeability $\mu$. Show that the field components for $H_{mn}$-waves are given by

$$H_z = C \cos K_1 x \cos K_2 y e^{i(\omega t - \beta z)}, \quad E_z = 0$$

$$H_x = -\frac{\beta c}{\omega\mu} E_y = \frac{Ci\beta K_1}{k^2} \sin K_1 x \cos K_2 y \, e^{i(\omega t - \beta z)}$$

$$H_y = \frac{\beta c}{\omega\mu} E_x = \frac{Ci\beta K_2}{k^2} \cos K_1 x \sin K_2 y \, e^{i(\omega t - \beta z)}$$

where $K_1 = \frac{m\pi}{a}$, $K_2 = \frac{n\pi}{b}$, $k^2 = K_1^2 + K_2^2$ and $\beta^2 c^2 = \omega^2 \mu K - k^2 c^2$

Find the distribution of current density $J$ and the distribution of surface charge $\sigma$ which are induced on the wall $x = 0$ by the $H_{mn}$ wave. Check your results by verifying that $\frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = -\dot{\sigma}$. (E.)

7. The regions $x < 0$, $0 < x < h$, $h < x$ are occupied by media defined respectively by constants ($K = 1$, $\mu = 1$, $\sigma = 0$), ($K$, $\mu = 1$, $\sigma = 0$), and ($K = 1$, $\mu = 1$, $\sigma$). A train of plane harmonic electromagnetic waves moving with constant velocity in the region $x < 0$ is incident normally on the plane $x = 0$. If the waves in the region $0 < x < h$ have wavelength $h/m$, where $m$ is an integer, show that the ratio of the squares of the amplitude of the incident electric wave in the region $x < 0$ and the transmitted electric wave in the region $h < x$ is given by $\frac{1}{4}\{(n + 1)^2 + p^2\}e^{-4m\pi p/\sqrt{K}}$, where $(n - ip)^2 = 1 - 2ih\sigma\sqrt{K}/m$. (N.)

8. For a non-conducting dielectric medium show briefly how Maxwell's equations can be expressed in terms of a vector potential **A** and a scalar potential $\phi$ and with the usual notation establish the equations

$$\mathbf{E} = -\frac{1}{c}\dot{\mathbf{A}} - \nabla\phi, \quad \operatorname{div} \mathbf{A} = \frac{-K\mu}{c}\dot{\phi}, \quad \nabla^2 \mathbf{A} = \frac{K\mu}{c^2}\ddot{\mathbf{A}}$$

Determine the cylindrical polar components of the field vector **E** in terms of $(r, \theta, z)$ if $\mathbf{A} = \mathbf{k} f(r) \exp 2\pi i(z/\lambda - t/\tau)$ where $\lambda$ and $\tau$ are constants. Such electromagnetic waves are propagated along a dielectric cylinder of constants $(K, \mu)$ with axis in the $z$-direction bounded by a perfectly conducting surface at $r = a$. Show that the condition

$$\frac{K\mu}{c^2\tau^2} = \frac{1}{\lambda^2} + \frac{b^2}{4\pi^2}$$

must be satisfied, where $b$ is a root of the equation $Y(ba) = 0$ and $Y(x)$ is the solution, finite at $x = 0$, of the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0 \qquad \text{(N.)}$$

9. A circuit $c$ carrying current $I$ is set in a medium of unit permeability in the presence of a magnetic field of intensity **H**. Show that the force exerted on the circuit is $\mathbf{F} = I\int_c d\mathbf{s} \times \mathbf{H}$.

A wire circuit in the form of an isosceles triangle $ABC$, right angled at $B$, carries a current $I_1$. An infinite straight wire $L$, parallel to $AB$, has a projection $L^1$ on the plane of $ABC$ intersecting $CB$ produced at $D$, where $CB = BD$. If $L$ carries a current $I_2$ and is at a distance equal to $BD$ from $L^1$, show that the force component acting on $ABC$ in a direction parallel to $BC$ is of magnitude $(\log \frac{5}{2} - 1)I_1 I_2$. (N.)

10. An infinite conducting slab of conductivity $\sigma$ is contained between the planes $z = \pm a$, and orthogonal rectilinear axes $Ox$, $Oy$ are located in the plane $z = 0$.

The current in the slab flows symmetrically with respect to the plane $z = 0$ parallel to the $x$-axis with density $J_x$ = real part of $e^{i\omega t}f(z)$ and the only non-zero magnetic intensity component is $H_y$. If, in the usual notation, curl $\mathbf{H} = 4\pi\mathbf{J}$, curl $\mathbf{E} = -\frac{1}{c}\dot{\mathbf{H}}$, div $\mathbf{E} = 0$, where $\mathbf{J} = c\sigma\mathbf{E}$, find the equation satisfied by $J_x$ and show that the total current flowing in the region contained by any two planes perpendicular to the $y$-axis at unit distance apart is the real part of

$$I = \frac{2J_0 e^{i\omega t}}{K} \sinh aK$$

where $J_0 = f(0)$ and $K^2 = 4\pi i\sigma\omega$. Determine also $E_x$ and $H_y$ on the plane $z = a$. (N.)

11. Define the vector potential of a magnetic field and obtain the expression

$$\mathbf{A}(\mathbf{r}) = \mu i \int \frac{ds^1}{|\mathbf{r} - \mathbf{r}^1|}$$

for the vector potential due to a steady current $i$ in a finite wire in a medium of permeability $\mu$.

Current $i$ flows along a wire which is in the shape of a curve whose parametric equations are

$$x = \frac{1 + t^4}{(1 + t^2)^2},\ y = \frac{2t}{(1 + t^2)^{3/2}},\ z = \frac{2t^3}{(1 + t^2)^2};\quad -\infty \leqslant t \leqslant \infty$$

Show that the vector potential at the origin is equal to the vector potential at an infinite distance from the wire. (D.)

12. A magnetic doublet of moment $m$ lies on the axis of a fixed circular wire of radius $a$ carrying a current $i$. If the axis of the doublet is along the axis of the wire and its distance from the wire is $x$, show that the doublet experiences a force of magnitude $6\pi ima^2x/(a^2 + x^2)^{5/2}$.

A fixed circular wire of radius $a$ has resistance $R$ and zero self-inductance. A magnetic doublet of moment $m$ is moving with velocity $v$ along its axis, which is also the axis of the wire. Find the magnetic flux through the wire, and hence show that the force opposing the motion of the doublet is

$$36\pi^2 m^2 v \sin^8\theta \cos^2\theta / Ra^4$$

where $\theta$ is the angle subtended at the doublet by a radius. (D.)

13. A transverse magnetic wave of the form

$$\mathbf{H} = \mathbf{H}_0\,(x, y)e^{i(pt-Kz)},\ \mathbf{E} = \mathbf{E}_0\,(x, y)e^{i(pt-Kz)}$$

where $p$ and $K$ are constants and $p$ is real, is propagated in a rectangular wave guide without dielectric, and having sides parallel to the $z$-axis. Prove that $E_z$ satisfies the equation

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \gamma^2 E_z = 0$$

where $\gamma^2 = p^2\mu_0\epsilon_0 - K^2$ in m.k.s. units.

For a guide having a square cross-section with sides of length $\pi$, the only component of the field in the direction of propagation is

$$E_z = \sin x \sin y\ e^{i(pt-Kz)}$$

Find the other components of the field and determine the "cut-off" frequency of the guide. (D.)

14. Show that the formulae $\mathbf{E}$ = curl curl $\mathbf{S}$, $\mathbf{H} = \frac{1}{c}$ curl $\left(\frac{\partial \mathbf{S}}{\partial t}\right)$ for the electric and magnetic vectors satisfy Maxwell's equations in empty space provided that in a rectangular coordinate system the vector function $\mathbf{S}$ is a solution of the wave equation $c^2\nabla^2\mathbf{S} = \frac{\partial^2\mathbf{S}}{\partial t^2}$.

Verify that $A\mathbf{k} \sin \alpha x \sin (\omega t - \gamma z)$ is a positive form of $\mathbf{S}$, where $A$, $\alpha$, $\gamma$ are real constants, $x$, $y$, $z$ rectangular coordinates and $\mathbf{k}$ is a unit vector along the $z$-axis, if $\alpha$, $\gamma$ and $\omega$ satisfy a certain condition.

Find the condition that such a solution should represent a field between two perfectly conducting planes $x = 0$, $x = a$ and prove that if the field is that of a wave propagated along the $z$-axis, then $\omega > n\pi c/a$, where $n$ is an integer.

Show also that the time average of the energy flow is parallel to the $z$-axis and of amount $W\gamma a\alpha^2 A^2/16\pi$ per unit width. (L.)

15. Show that in a homogeneous, uncharged, non-conducting medium, Maxwell's equations have a solution in which every component of the field vectors $\mathbf{E}$ and $\mathbf{H}$ vanishes except $E_x$ and $H_y$, and that such a solution corresponds to a plane wave travelling in the $z$-direction with velocity $c/\sqrt{\mu K}$, where $K$ is the dielectric constant and $\mu$ the permeability.

Prove that if a plane-polarised plane wave of monochromatic light travelling *in vacuo* falls normally on a slab of dielectric that occupies the region $z > 0$, the amplitudes of the electric intensity in the incident, reflected and transmitted waves are in the ratios

$$1 + \left(\frac{K}{\mu}\right)^{1/2} : 1 - \left(\frac{K}{\mu}\right)^{1/2} : 2 \qquad \text{(L.)}$$

16. A rectangular circuit is free to rotate about an axis bisecting two opposite sides; the sides parallel to the axis are of length $l$, and the sides bisected by the axis are of length $2a$. The axis is parallel to, and at a distance $b$ from, a long straight wire carrying a current $i$. Find the magnetic flux through the rectangle when its plane makes an angle $\theta$ with the plane containing the long straight wire and the axis.

Hence, or otherwise, find the couple acting on the rectangular circuit if it carries a current $i^1$ showing in particular that when $\theta = \pi/2$ the couple is $4ablii^1/(a^2 + b^2)$. (H.)

17. The magnetic scalar potential $\phi$ at a point $P$ due to a current $I$ flowing in a closed linear conductor $C$ is $\phi = I\Omega$, where $\Omega$ is the solid angle subtended by $C$ at $P$. Deduce the Biot–Savart law for the magnetic intensity $\mathbf{H}$ in free space due to $I$ in the form $\mathbf{H} = I\int_C \frac{d\mathbf{s} \times \mathbf{r}}{r^3}$ where $\mathbf{r}$ is the positive vector of $P$ relative to the element $d\mathbf{s}$ of $C$.

Two circular loops of wire each of radius $a$ have their planes perpendicular to $Oz$ and their centres at $(0, 0, b)$ and $(0, 0, -b)$, where $b > 0$. Equal currents $I$ flow through the wires in opposite senses, that in the wire for which $z$ is positive being in the positive sense relative to $Oz$. Prove that the magnetic field at a point $(x, y, z)$ near the origin is approximately $(-\frac{1}{2}\lambda x, -\frac{1}{2}\lambda y, \lambda z)$ where

$$\lambda = 12\pi I a^2 b/(a^2 + b^2)^{5/2} \qquad \text{(H.)}$$

18. Current $j$ flows in a circular coil of wire of radius $a$. Find the magnetostatic potential at a point $P$ on the axis of symmetry of the coil at a distance $z$ from the centre, and show that the magnetic field at $P$ has magnitude $2\pi j a^2/(a^2 + z^2)^{3/2}$.

Current $j$ flows through a closely wound cylindrical solenoid having circular cross-section and $n$ turns per unit length. A point $P$ on the axis is such that the angles subtended at $P$ by diameters of the two end circuits are $2\alpha$ and $2\beta$. Show that the magnetic field at $P$ is directed along the axis and find its magnitude. Compare this problem with that of finding the field produced by a uniform cylindrical bar magnet. (H.)

19. A current $I$ flows in a circular wire of radius $a$ and centre $O$. Show that the magnetic field at a point $P$ on the axis of the wire is $2\pi a^2 I/R^3$, where $R$ is the distance of $P$ from any point on the wire.

A circular coil of radius $2a$, coplanar and concentric with the wire, is added. It has eight turns, and the same current $I$ circulates in it in the opposite sense to that

in the wire radius $a$. Show that the magnetic field at $O$ is numerically three times as strong as it was before, and that the field at points on the axis at a small distance $x$ from $O$ is nearly uniform, the variation from uniformity being of order $x^4/a^4$.

(H.)

20. A thin vertical bar of iron is magnetised by an alternating current so that the strength of its poles are $\pm m \sin pt$. A horizontal circular wire, of radius $a$, inductance $L$, and resistance $R$, is placed with its centre on the bar so that its radius subtends angles $\theta$, $\theta^1$ at the poles. Find the current induced in the wire and show that the wire will experience a mean force

$$-\frac{2\pi^2 m^2 p^2 L}{a(R^2 + L^2p^2)}(\sin^3\theta - \sin^3\theta^1)(2 - \cos\theta - \cos\theta^1) \qquad \text{(L.)}$$

21. In a certain electromagnetic field in a non-conducting dielectric of constant $K$ the scalar potential is zero and the vector potential is $A_x = 0$, $A_y = 0$, $A_z = f(r)e^{i\alpha(z-nt)}$, where $r^2 = x^2 + y^2$. Determine the cylindrical polar components of the electric and magnetic intensities **E** and **H**.

Show that if such electromagnetic waves are propagated along a dielectric cylinder with a conducting boundary $r = a$, the speed of propagation is $u(1 + s^2/\alpha 2)^{1/2}$, where $u$ is the speed of light in the dielectric and $s$ is a root of the equation $J_0(sa) = 0$.

(O.)

22. A circular loop of radius $a$, mass $M$, resistance $R_1$ and inductance $L$ has its centre at distance $x$ from a long straight wire carrying a constant current $I$ e.s.u. The wire and loop are in a plane and the medium is of unit permeability. Show that the magnetic flux $\Phi$ through the loop due to the current $I$ is

$$\frac{4\pi I}{c}\left\{x - \sqrt{x^2 - a^2}\right\}$$

Initially $x = x_0$ and the loop is given a velocity $u$ in its plane perpendicular to and away from the wire. Write down the equation for the current $J$ in the loop at any time. Given that the magnetic force on the loop is $(J/c)\,(d\Phi/dx)$, verify that the rate of dissipation of energy in the loop through resistance is equal to the rate of decrease of the sum of the kinetic energy and the magnetic self-energy of the loop.

Derive an equation for the distance $x$, at which the loop comes to rest, assuming that self-induction can be neglected. (C.)

# CHAPTER 6

# WATER WAVES

## 6.1 Basic Assumptions

The basic hydrodynamic equations (Chapter 3) can be modified for the special case of wave motion, i.e. of *small* disturbances in a fluid. We assume first that

(1) The velocities are small so that second order terms may be neglected. The equations (3.1) then reduce to

(*a*) continuity (incompressible fluid) $\quad \operatorname{div} \mathbf{q} = 0$

(*b*) acceleration $\quad \mathbf{f} = \dfrac{\partial \mathbf{q}}{\partial t}$

(*c*) equation of motion $\quad \dfrac{\partial \mathbf{q}}{\partial t} = \mathbf{E} - \dfrac{1}{\rho} \operatorname{grad} p$

and Bernouilli's equation (3.1.1) reduces (for irrotational motion) to

(*d*) $$V + p/\rho - \frac{\partial \phi}{\partial t} = \text{constant}$$

where the two terms $\dfrac{\partial \phi}{\partial t} + G(t)$ in (3.1.1) have been replaced by $\dfrac{\partial \phi}{\partial t} +$ constant by adding a term in $t$ to the original $\phi$. This is permissible, since this term is constant in space and so does not affect the space derivatives of $\phi$.

Conditions (3.1.2), (3.2) are unchanged, and so

(*e*) at any boundary the normal velocities of fluid and boundary are equal;

(*f*) in irrotational motion $\nabla^2\phi = 0$

Also

(*g*) for two dimensional motion—waves in one direction only—there exists a stream function $\psi$, constant along a streamline, and satisfying $\nabla^2\psi = 0$ when the motion is also irrotational. (See Vol. II.)

### **6.1.1 Notation** (*a*) Waves in One Direction

The standard system which will be used in the problems will be to take the undisturbed free surface as $y = 0$ and the $y$-axis upwards, the $x$-axis in the direction of wave propagation. The shape of the surface

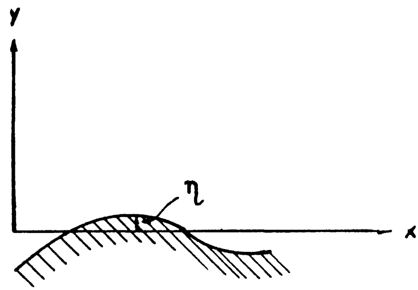


Fig. 51

at any instant will be defined by $y = \eta$ where $\eta$ is, in general, a function of $x$ and $t$.

(*b*) General Wave Motion

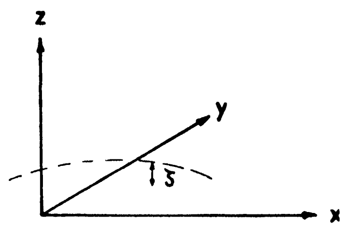


Fig. 52

In this case the undisturbed free surface is taken as the plane $z = 0$ with the $z$-axis upwards, and the wave motion is defined by $z = \zeta$ where $\zeta$ is, in general, a function of $x$, $y$, and $t$.

Most of the problems which follow are two-dimensional so that the axes in (*a*) are used.

### 6.2 Surface Waves

These are waves on deep water satisfying the condition:

(2) The wave profile moves with the liquid, i.e. the velocities of wave and liquid are the same at the liquid surface.

This gives the additional condition

(*h*) $\qquad \dfrac{\partial\eta}{\partial t} = -\dfrac{\partial\phi}{\partial y}$ at the liquid surface: or $\dfrac{\partial\zeta}{\partial t} = -\dfrac{\partial\phi}{\partial z}$

Such waves are in general irrotational, and are governed by the set of conditions (*a*)–(*h*).

Some commonly occurring special cases may be listed. (The conditions appropriate to the axes 6.1.1 (*a*) are quoted, unless otherwise stated.)

**6.2.1 Liquid under Gravity only.** In this case the external force $\mathbf{F}$ is $(0, -g)$, the potential $V$ is $gy$, and the pressure $p = p_0$, constant, at the liquid surface $y = \eta$. Hence the conditions become

$$(c)\qquad \frac{\partial \mathbf{q}}{\partial t} = \mathbf{F} - \frac{1}{\rho}\operatorname{grad} p$$

$(d)\ g\eta - \dfrac{\partial \phi}{\partial t} = 0$ at the liquid surface (defining $\phi$ so that the constant vanishes)

Also $gy + p/\rho - \dfrac{\partial \phi}{\partial t} = p_0/\rho$ anywhere in the liquid.

$(h)\ \dfrac{\partial \eta}{\partial t} = -\dfrac{\partial \phi}{\partial y}$ at the liquid surface; or

$(g)$ Stream function $\psi$ constant at liquid surface together with $(e)$ normal velocities of fluid and boundaries equal; or $(g)$ stream function $\psi$ constant on fixed boundary.

$$(f)\qquad \nabla^2\phi = 0$$

It should be noted that $(d)$ and $(g)$ may be combined to eliminate $\eta$ and give $(i)$

$$g\frac{\partial \phi}{\partial y} + \frac{\partial^2 \phi}{\partial t^2} = 0$$

at the liquid surface.

**6.2.2 Interface between Two Liquids.** In this case there exist potentials $\phi, \phi^1$ in the two liquids (densities $\rho, \rho^1$), and the conditions which hold at the interface are that both the pressure $p$ and the vertical velocity must be continuous there. Hence the corresponding set of conditions is

$$\left.\begin{array}{l} (d)\ \rho\left(V - \dfrac{\partial \phi}{\partial t}\right) \text{ continuous} \\[2ex] (h)\ \dfrac{\partial \phi}{\partial y} = \dfrac{\partial \phi^1}{\partial y} = -\dfrac{\partial \eta}{\partial t} \end{array}\right\} \text{ at the interface}$$

with $(e)$ normal velocities of liquids and boundaries equal
or $(g)$ stream function $\psi$ constant on a fixed boundary,

and $(f)$

$$\nabla^2\phi = \nabla^2\phi^1 = 0$$

If gravity is the only external force, then, as above, $V = gy$, and eliminating $\eta$ from $(d)$ and $(h)$ gives

$$(i)\qquad \rho\left(\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial y}\right) = \rho^1\left(\frac{\partial^2 \phi^1}{\partial t^2} + g\frac{\partial \phi^1}{\partial y}\right)$$

**6.2.3 Surface Tension at Liquid Surface.** When surface tension $T$ is taken into account, the pressure $p_1$ just inside the liquid is smaller than the

outside pressure $p_0$ by $T/R$, where $R$ is the radius of curvature of the liquid surface, so that

$$p_1 = p_0 - T\frac{\partial^2\eta}{\partial x^2} \qquad \text{(2 dimensions)}$$

or

$$p_1 = p_0 - T\left(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2}\right) \quad \text{(3 dimensions)}$$

Thus condition ($d$) becomes

$$\left.\begin{array}{l} (d)\ \text{(2 dimensions)}\ \dfrac{\partial\phi}{\partial t} - V + T\dfrac{\partial^2\eta}{\partial x^2} = 0 \\ \text{(3 dimensions)}\ \dfrac{\partial\phi}{\partial t} - V + T\left(\dfrac{\partial^2\zeta}{\partial x^2} + \dfrac{\partial^2\zeta}{\partial y^2}\right) = 0 \end{array}\right\} \text{at the free surface}$$

**6.2.4 Stationary Waves, Steady Motion.** It is possible to have a system where the wave profile remains unchanging in space while the fluid flows along it (see 6.4 (iii)). The conditions in 6.2.1–6.2.3 which depend on time variation no longer apply. The corresponding conditions are then

$$\text{Bernouilli's equation, } V + p/\rho + \tfrac{1}{2}q^2 = \text{constant}$$

(terms in speed can no longer be neglected, since there is a streaming motion) which, for a liquid under gravity only, gives

($d$) $g\eta + \frac{1}{2}q^2 = \text{constant}$ at the liquid surface;

also ($h$) is replaced by

($j$) $y = \eta$ is a stream line (or $z = \zeta$ a stream surface)

i.e. $$\frac{\partial\phi}{\partial n} = 0,\ \psi = \text{constant on } y = \eta \text{ (or on } z = \zeta)$$

In addition, there are as usual the conditions

($e$) normal velocities of fluid and boundaries equal (or $\psi$ = constant on a fixed boundary)

($f$) $\nabla^2\phi = 0$

## 6.3 Long or Tidal Waves

These are defined by the condition, replacing 6.2 (2), that:

(2) the wavelength is much greater than the depth, which implies that the vertical acceleration may be neglected and that the horizontal acceleration is the same at all depths.

In this case the wave profile does *not*, in general, move with the liquid, and the motion is *not* necessarily irrotational.

**6.3.1 Liquid under Gravity only.** The equation of motion 6.1 ($c$) gives, using condition (2) above,

$$\frac{\partial p}{\partial y} = -g\rho$$

i.e. ($c$) $$p = p_0 + g\rho(\eta - y),$$

$p_0$ being the pressure at the free surface $y = \eta$.

Substituting this value of $p$ into 6.1 ($c$) gives

$$(c)\begin{cases}\dfrac{\partial q_1}{\partial y}=0, \dfrac{\partial q_1}{\partial t}=-g\dfrac{\partial \eta}{\partial x} & \text{(2 dimensions)}\\[2ex] \dfrac{\partial q_1}{\partial z}=0, \dfrac{\partial q_2}{\partial z}=0 \text{ and } \dfrac{\partial q_1}{\partial t}=-g\dfrac{\partial \zeta}{\partial x}, \dfrac{\partial q_2}{\partial t}=-g\dfrac{\partial \zeta}{\partial y} & \end{cases}$$

(3 dimensions)

Since (6.1 ($e$)) $q_1 = 0$ on a boundary $x =$ constant, $q_2 = 0$ on $y =$ constant, it follows from ($c$) that $\partial\zeta/\partial x = 0$ on $x =$ constant, $\partial\zeta/\partial y = 0$ on $y =$ constant, and by extension, $\dfrac{\partial \zeta}{\partial n} = 0$ on any boundary $lx + my = 1$. Also on the base the vertical velocity is zero.

Since the horizontal acceleration is the same at all depths, so is the horizontal velocity, and an equation of continuity ($a$) may be given directly connecting the velocity with changes in depth or breadth of the channel.

## 6.4 Characteristics of Wave Motion in One Direction

(i) *Progressive Waves on Still Liquid.* A simple harmonic progressive wave in the $x$-direction is represented by a surface elevation

$$\eta = a \sin (mx - nt).$$

This gives a motion in which the wave profile $\eta = a \sin mx$ moves parallel to the $x$-axis with velocity $c = n/m$ the *velocity* of *propagation*

$a$ is called the *amplitude* of the wave.

$\lambda = 2\pi/m$ is the *wavelength*, i.e. distance between successive crests.

$\tau = 2\pi/n$ is the *waveperiod*, i.e. interval between successive times at which the elevation of a point on the free surface is the same.

$\dfrac{1}{\tau} = n/2\pi$ is the *wave frequency*.

(ii) *Stationary or Standing Waves on Still Liquid.* A simple harmonic stationary wave in the $x$-direction is represented by a surface elevation $\eta = a \sin mx \cos nt$.

This gives a motion in which the wave profile $\eta = a \sin mx$ is multiplied as a whole by the factor $\cos nt$ varying from $+1$ to $-1$.

The effect is that crests and troughs alternate with each other, while the nodes—the points where $\eta = 0$—remain unchanged.

(iii) *Stationary Waves on a Steady Stream.* A steadily flowing stream can carry waves which are stationary in a different sense from the stationary waves in (ii). Such waves have a profile constant in space, an effect which may be produced by combining a progressive wave with a stream of equal and opposite velocity.

(iv) *Group Velocity of a Train of Waves.* If the overall surface

elevation is due to the superposition of a train of progressive waves of slightly differing wavelengths, the *group* velocity $c_g$ is given by

$$c_g = \frac{dn}{dm} = c + m\frac{dc}{dm} = c - \lambda\frac{dc}{d\lambda} \text{ where } c = n/m$$

(v) *Energy.* The *energy* of a harmonic wave is the energy of the liquid between surfaces a wavelength apart.

For a progressive wave, in a channel of constant depth and breadth,

$$\text{K.E.} = \text{P.E.} = \tfrac{1}{4}a^2\rho g\lambda$$

For a stationary wave under the same conditions

$$\begin{aligned}\text{K.E.} &= \tfrac{1}{4}a^2\rho g\lambda \sin^2 nt \\ \text{P.E.} &= \tfrac{1}{4}a^2\rho g\lambda \cos^2 nt\end{aligned}$$

since in this case the liquid is instantaneously at rest when the wave crests turn, i.e. when $\cos nt = \pm 1$.

The problems which follow deal mainly with surface waves under different conditions; Problem 81, however, is on tidal waves in a shelving canal.

---

**Problem 75**

(i) Show that the velocity potential for a train of straight-crested waves of small elevation $\eta = a \sin (mx - nt)$ on liquid of density $\rho$ and uniform depth $h$ is

$$\phi = \frac{ga}{n} \frac{\cosh m(y + h)}{\cosh mh} \cos (mx - nt)$$

and that $\qquad n^2 = mg \tanh mh$

(ii) Show that the time average of the kinetic energy of a vertical column of this liquid is $\frac{1}{2}E = \frac{1}{4}\rho g a^2$ per unit area of the base of the column; and that this is equal to the time average of the potential energy of the column.

(iii) Show that the time average of the rate at which work is done by the fluid pressure across a vertical strip of the plane $x =$ constant is, per unit width of the strip,

$$\tfrac{1}{4}\rho g a^2 c\left(1 + \frac{mh}{\sinh mh \cosh mh}\right) = E\frac{d(mc)}{dm}$$

where $c = n/m$ is the wave velocity.

(iv) Assuming the above results to hold for a group of waves of slowly varying amplitude, deduce that the group travels with velocity

$$C = \frac{d(mc)}{dm} \qquad \text{(O.)}$$

---

**Solution.** The conditions applicable are those for surface waves on a liquid under gravity, i.e. the conditions 6.2.1 (*c*)–(*h*) or (*i*).

(i) We are given the elevation $\eta$, and hence find

$$\left.\begin{aligned} &\text{from 6.2.1 } (d) \quad \frac{\partial\phi}{\partial t} = ga\sin(mx - nt) \\ &\text{from 6.2.1 } (h) \quad \frac{\partial\phi}{\partial y} = na\cos(mx - nt) \end{aligned}\right\} \text{on the free surface } y = 0 \quad . \quad (1)$$

and since the canal is of uniform depth $h$,

$$\text{from 6.2.1 } (e) \qquad \frac{\partial\phi}{\partial y} = 0 \text{ at the base, } y = -h \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{and from 6.2.1 } (f) \qquad \nabla^2\phi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

By inspection of (1), $\phi$ must be of the form $f(y)\cos(mx - nt)$, and substitution in (3) leads to

$$-m^2 f(y) + f''(y) = 0$$

i.e.
$$f(y) = Ae^{my} + Be^{-my}$$

while substitution of this in (1) and (2) gives finally

$$\phi = \frac{ga}{n\cosh mh}\cosh m(y + h)\cos(mx - nt) \quad . \quad . \quad (4)$$

where
$$n^2 = mg\tanh mh \quad . \quad . \quad . \quad . \quad . \quad (5)$$

(ii) The kinetic energy of a vertical column of unit base area is $\int \frac{1}{2}\rho q^2\,dv = \frac{1}{2}\rho\int \phi\frac{\partial\phi}{\partial n}\,dS$ by the usual transformation. Over the base of the column $\frac{\partial\phi}{\partial n} = 0$ and over the sides $\frac{\partial\phi}{\partial n} = \pm\frac{\partial\phi}{\partial x}$ while over the top $y = 0$, to a first approximation, $\frac{\partial\phi}{\partial n} = \frac{\partial\phi}{\partial y}$.

Hence K.E. =

$$\tfrac{1}{2}\rho\frac{g^2a^2}{n^2\cosh^2 mh}\Bigg[\int_{-h}^{0}\cosh^2 m(y + h)\{m\sin(mx_0 - nt)\cos(mx_0 - nt) - m\sin(mx_1 - nt)\cos(mx_1 - nt)\}dy + \int_{x_0}^{x_1} m\cos^2(mx - nt)\cosh mh\sinh mh\,dx\Bigg]$$

where $x_1 - x_0 = 1$ unit.

Now the time average of $\sin(mx - nt)\cos(mx - nt)$ is zero for any $x$, so the first integral has an average of zero and the K.E. is given by the time average of the second integral,

$$\tfrac{1}{4}\frac{\rho g^2a^2}{n^2}m\tanh mh$$

$$= \tfrac{1}{4}\rho g a^2 \text{ using (5) above}$$

The potential energy of the same vertical column is due to the elevation of columns of liquid to height $\eta$, i.e. is

$$\int_{x_0}^{x_1} \tfrac{1}{2}\rho g \eta^2 \, dx$$

which substituting $\eta = a \sin (mx - nt)$ and then taking the time average, becomes $\frac{1}{4}\rho g a^2$.

If the total energy is $E$, then

$$\text{K.E.} = \text{P.E.} = \tfrac{1}{2}E = \tfrac{1}{4}\rho g a^2 \quad . \quad . \quad . \quad . \quad (6)$$

(iii) The rate of doing work across a vertical strip, unit width, is

$$\frac{dW}{dt} = \int_{-h}^{0} p q_1 \, dy \text{ where } p = p_0 + \rho \frac{\partial \phi}{\partial t} - g\rho y, \; q_1 = -\frac{\partial \phi}{\partial x}$$

Substituting for $\phi$, this gives

$$\frac{dW}{dt} = \sin (mx - nt) \int_{-h}^{0} \frac{mga \cosh m(y + h)(p_0 - g\rho y)}{n \cosh mh} dy$$

$$+ \sin^2 (mx - nt) \int_{-h}^{0} \frac{m\rho n g^2 a^2 \cosh^2 m(y + h)}{n^2 \cosh^2 mh} dy$$

and integrating and averaging over time, since

$$\text{average } \sin^2 (mx - nt) = \tfrac{1}{2},$$

gives mean
$$\frac{dW}{dt} = \tfrac{1}{4} \frac{m\rho g^2 a^2}{n \cosh^2 mh}\left(h + \frac{1}{2m} \sinh 2mh\right)$$

Using (5), (6) and writing $c = n/m$, the wave propagation velocity, gives

$$\text{mean } \frac{dW}{dt} = \tfrac{1}{2}Ec\left(1 + \frac{mh}{\sinh mh \cosh mh}\right) \quad . \quad . \quad (7)$$

Differentiating (5) gives

$$2n \frac{dn}{dm} = g \tanh mh + mgh \operatorname{sech}^2 mh$$

i.e.
$$\frac{dn}{dm} = \tfrac{1}{2}c\left(1 + \frac{mh}{\sinh mh \cosh mh}\right)$$

Thus the mean rate of doing work from (7) is given by

$$E\frac{dn}{dm} = E\frac{d(mc)}{dm}$$

(iv) If these results are assumed to hold for a group of waves of slightly varying amplitude, then for such a group:

mean total energy $= E$

mean rate of doing work $= E\dfrac{d(mc)}{dm}$ = rate of energy change

Thus, the rate at which energy is transferred along the wave direction is $d(mc)/dm$, and this must be the rate at which the *group* of waves travels.

---

*Alternatively.* Since the motion is two-dimensional, we may use a complex potential $w = \phi + i\psi$.

The conditions are from (1) that the real part of $w$ satisfies

$$\left.\begin{aligned} \frac{\partial\phi}{\partial t} &= ga \sin (mx - nt) \\ \frac{\partial\phi}{\partial y} &= na \cos (mx - nt) \end{aligned}\right\} \text{ on } y = 0$$

and from 6.2.1 ($g$) the imaginary part of $w$ is constant on $y = -h$.

To fit the conditions, consider

$$w = A \cos \{m(z + ih) - nt\}$$

This satisfies (2), the streamline condition at the base of the channel, since imaginary part of $w = 0$ on $y = -h$, and application of conditions (1) at $y = 0$ gives

$$An \cosh mh = ga$$
$$Am \sinh mh = na$$

from which

$$w = \frac{ga}{n \cosh mh} \cos \{m(z + ih) - nt\}$$

and $\quad n^2 = mg \tanh mh$ as required.

The problem has then to be completed as before.

---

**Problem 76**

Liquids of densities $\rho$ and $\rho^1$ ($\rho < \rho^1$) are confined between rigid plane boundaries $y = h$ and $y = -h^1$, the plane $y = 0$ being the surface of separation of the liquids. Waves of wavelength $2\pi/m$ are propagated along the interface $y = 0$ of the two liquids. Show that the velocity of propagation is given by

$$c^2 = g(\rho^1 - \rho)\{m(\rho \coth mh + \rho^1 \coth mh^1)\}^{-1} \qquad \text{(L.)}$$

---

**Solution.** We seek potentials $\phi$, $\phi^1$ satisfying conditions 6.2.2 ($d$)–($h$).

Since the waves are to have wavelength $2\pi/m$, a suitable choice satisfying ($f$) will be, as in problem 75

$$\phi = (Ae^{my} + Be^{-my}) \cos (mx - nt)$$
$$\phi^1 = (A^1e^{my} + B^1e^{-my}) \cos (mx - nt)$$

and to satisfy (*e*) these become

$$\left.\begin{aligned}\phi &= 2Ae^{mh}\cosh m(h-y)\cos(mx-nt)\\ \phi^1 &= 2A^1e^{mh^1}\cosh m(y+h^1)\cos(mx-nt)\end{aligned}\right\} \quad . \quad . \quad (1)$$

Then (*h*) gives $A^1e^{-mh^1}\sinh mh^1 = -Ae^{mh}\sinh mh$ . . . (2)
and (*i*) gives

$$\rho^1A^1e^{-mh^1}\sinh mh^1(-\sigma^2\coth mh^1 + mg)$$
$$= \rho Ae^{mh}\sinh mh(-\sigma^2\coth mh - mg) \quad (3)$$

From (2) and (3) we obtain

$$n^2 = \frac{mg(\rho-\rho^1)}{\rho^1\coth mh^1 + \rho\coth mh}$$

and hence the wave velocity $c$, where $c = n/m$, is given by

$$c^2 = \frac{g(\rho-\rho^1)}{m(\rho^1\coth mh^1 + \rho\coth mh)}$$

---

**Problem 77**

A stream of depth $h$ is flowing with velocity $V$, and on its surface there are stationary waves of small amplitude and of wavelength $\lambda$. Neglecting the effects of surface tension, prove that

$$V^2 = \frac{g\lambda}{2\pi}\tanh\frac{2\pi h}{\lambda}$$

and deduce that $V$ must be less than $\sqrt{gh}$.

Find the corresponding equation when account is taken of the surface tension $T$ and the stream is assumed to be very deep. Deduce that there exists a critical value of $V$ below which the existence of surface waves is impossible. (L.)

---

**Solution.** The conditions are given in 6.2.4 (*d*), (*e*), (*f*), and (*j*). Suppose the fixed profile to be $y = \eta = a\sin 2\pi x/\lambda$. Assume potential

$$\phi = Vx + f(y)\cos 2\pi x\lambda$$

where the conditions (*f*) $\nabla^2\phi = 0$ and (*e*) $\dfrac{\partial\phi}{\partial y} = 0$ on $y = 0$ give

$$\phi = Vx + b\cosh\frac{2\pi}{\lambda}(y+h)\cos\frac{2\pi x}{\lambda} \quad . \quad . \quad . \quad (1)$$

The corresponding stream function is

$$\psi = Vy - b\sinh\frac{2\pi}{\lambda}(y+h)\sin\frac{2\pi x}{\lambda} \quad . \quad . \quad . \quad (2)$$

Here $a$ and $b$ are both small quantities.

Condition ($j$), that $y = a \sin 2\pi x/\lambda$ is a streamline, gives, substituting in (2),

$$\text{constant} = Va \sin 2\pi x/\lambda - b \sinh 2\pi h/\lambda \sin 2\pi x/\lambda$$

(neglecting the product $ab$), and so

$$Va - b \sinh 2\pi h/\lambda = 0 \qquad . \quad . \quad . \quad . \quad (3)$$

and condition ($d$) gives, at the liquid surface ($y = 0$ to a first approximation),

$$ga \sin 2\pi x/\lambda + \tfrac{1}{2}\left\{\left(V - \frac{2\pi b}{\lambda}\cosh\frac{2\pi h}{\lambda}\sin\frac{2\pi x}{\lambda}\right)^2 + \left(\frac{2\pi b}{\lambda}\sinh\frac{2\pi h}{\lambda}\cos\frac{2\pi x}{\lambda}\right)^2\right\} = \text{constant}$$

i.e. neglecting second order quantities,

$$ga \sin 2\pi x/\lambda + \tfrac{1}{2}\left(V^2 - \frac{4\pi Vb}{\lambda}\cosh\frac{2\pi h}{\lambda}\sin\frac{2\pi x}{\lambda}\right) = \text{constant}$$

which means

$$ga - \frac{2\pi Vb}{\lambda}\cosh\frac{2\pi h}{\lambda} = 0 \qquad . \quad . \quad . \quad . \quad (4)$$

Eliminating $a$ and $b$ from (3) and (4) gives

$$V^2 = \frac{g\lambda}{2\pi}\tanh\frac{2\pi h}{\lambda} \quad \text{as required} \quad . \quad . \quad . \quad . \quad (5)$$

The maximum value of $V^2$ with respect to $\lambda$ is found when $\dfrac{d(V^2)}{d\lambda} = 0$,

i.e. when $$\tanh\frac{2\pi h}{\lambda} = \frac{2\pi h}{\lambda}\operatorname{sech}^2\frac{2\pi h}{\lambda}$$

This means $$\tanh\frac{2\pi h}{\lambda} < \frac{2\pi h}{\lambda}$$

and so, from (5), $$V^2 < gh.$$

When the effect of surface tension is considered also, the only change is that (4) is replaced by (from 6.2.3 ($d$) applied to these conditions)

$$ga \sin\frac{2\pi x}{\lambda} + \tfrac{1}{2}\left(V^2 - \frac{4\pi Vb}{\lambda}\cosh\frac{2\pi h}{\lambda}\sin\frac{2\pi x}{\lambda}\right) + \frac{Ta}{\rho}\left(\frac{2\pi}{\lambda}\right)^2\sin\frac{2\pi x}{\lambda} = \text{constant}$$

i.e. $$ga - \frac{2\pi Vb}{\lambda}\cosh\frac{2\pi h}{\lambda} + \frac{4\pi^2 Ta}{\rho\lambda^2} = 0 \quad . \quad . \quad . \quad (6)$$

Eliminating $a$ and $b$ from (3) and (6) gives

$$V^2 = \left(g + \frac{4\pi^2 T}{\rho\lambda^2}\right)\frac{\lambda}{2\pi}\tanh\frac{2\pi h}{\lambda}$$

and if the stream is very deep, so that $\tanh\dfrac{2\pi h}{\lambda} \longrightarrow 1$, then

$$V^2 = \left(g + \frac{4\pi^2 T}{\rho\lambda^2}\right)\frac{\lambda}{2\pi} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This is the corresponding equation to (5) for a deep stream with surface tension $T$.

Inspection of (7) shows that $V^2 \longrightarrow \infty$ as $\lambda \longrightarrow 0$ and as $\lambda \longrightarrow \infty$ and that $V^2$ has one turning value, clearly a minimum, for positive $\lambda$. This value is given by

$$d(V^2)/d\lambda = 0$$

which gives

$$\lambda = 2\pi\sqrt{\frac{T}{g\rho}}$$

and, by substitution in (7),

$$V^2 = 2\sqrt{\frac{Tg}{\rho}}$$

This is the critical value of $V$ below which stationary surface waves are impossible.

---

**Problem 78**

A uniform incompressible liquid of negligible viscosity and surface tension flows with mean velocity $U$ over a corrugated surface whose equation is $y = (\varepsilon/K)\sin Kx$, $y$ being measured upwards and $\varepsilon$ small compared to unity. The mean depth is $h$, and the direction of mean flow is parallel to the $x$-axis. Prove that in steady flow the equation of the free surface is

$$y = h + (\varepsilon/K)\left(\cosh Kh - \frac{g}{KU^2}\sinh Kh\right)^{-1}\sin Kx \qquad \text{(C.)}$$

---

**Solution.** The form of the equation for the base indicates that the origin has been taken at the mean base level instead of in the free surface.

The requirement is for steady flow and so the conditions are those in 6.2.4 ($d$), ($e$), ($f$), and ($j$) as in the previous problem.

We assume a potential $\phi = Ux + f(y)\cos Kx$ assuming that since the corrugations have wavelength $2\pi/K$, so will a *steady* displacement due to them.

To satisfy ($f$) $\phi$ must be of the form

$$\phi = Ux + (a\cosh Ky + b\sinh Ky)\cos Kx \quad . \quad . \quad (1)$$

$a$, $b$ being small (of order $\varepsilon$), since they represent the disturbance due to the corrugations.

As in problem 77, it is easiest to proceed by using the stream function, which is, from $\phi$,

$$\psi = Uy - (a \sinh Ky + b \cosh Ky) \sin Kx \quad . \quad . \quad (2)$$

We are now not given the form for the stationary wave profile, and so we assume that the surface is

$$y = h + \eta = \eta = h + c \sin Kx \quad . \quad . \quad . \quad . \quad (3)$$

where $c$ is small (of order $\varepsilon$).

Then there are three conditions to fix the three constants $a$, $b$, $c$, namely streamline conditions on the base ($e$) and on the surface ($j$) and also the condition of constant pressure ($d$).

From ($e$)

$$\frac{U\varepsilon}{K} \sin Kx - a \sinh (\varepsilon \sin Kx) \sin Kx - b \cosh (\varepsilon \sin Kx) \sin Kx = \text{constant}$$

i.e. to first order in $\varepsilon$, $\qquad b = U\varepsilon/K \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$

From ($j$) $\quad U(h + c \sin Kx) - a \sinh (Kh + Kc \sin Kx) - b \cosh (Kh + Kc \sin Kx) \sin Kx = \text{constant}$

i.e. to first order in $\varepsilon$,

$$Uc = a \sinh Kh + b \cosh Kh \quad . \quad . \quad . \quad . \quad (5)$$

From ($d$)

$$g(h + c \sin Kx) + \tfrac{1}{2}[\{U - K(a \cosh Kh + b \sinh Kh) \sin Kx\}^2 + K^2(a \sinh Kh + b \cosh Kh)^2 \cos^2 Kx]$$

$= \text{constant}$ (putting $y = h$ for the free surface in the small terms)

and, to first order in $\varepsilon$ this reduces to

$$gc = UK(a \cosh Kh + b \sinh Kh) \quad . \quad . \quad . \quad (6)$$

Solving (4), (5), and (6) gives

$$c = (\varepsilon/K)\{\cosh Kh - (g/KU^2) \sinh Kh\}^{-1}$$

and so the surface equation is

$$y = h + (\varepsilon/K)\{\cosh Kh - (g/KU^2) \sinh Kh\}^{-1} \sin Kx$$

as required.

---

**Comment.** In these two problems it is necessary to approximate at each step, since the overall approximations, ignoring $q$ and equating the expressions for upward velocity at the free surface, no longer hold.

The use of the stream function is not absolutely essential, but makes the streamline condition much easier to use; the alternative would be to use $\partial\phi/\partial n = 0$, but the approximation $\partial/\partial n = \partial/\partial y$ is *not* applicable at the free surface, as can be seen by comparing $\dfrac{\partial\phi}{\partial y}$ from (1) with (5)

---

**Problem 79**

The free surface of water of uniform depth $h$ is disturbed by standing waves of elevation $a \sin(2\pi x/\lambda)\cos(2\pi vt/\lambda)$, where $a$ is small compared with $h$. Show that the velocity potential of the motion of the water, assumed irrotational, is given by

$$\phi = \frac{av \sin(2\pi x/\lambda)\sin(2\pi vt/\lambda)\cosh\{2\pi(y+h)/\lambda\}}{\sinh(2\pi h/\lambda)}$$

and that the velocity $v$ is given by

$$v^2 = \left(\frac{g\lambda}{2\pi}\right)\tanh(2\pi h/\lambda)$$

Show further that the paths of the particles of water are straight lines, described harmonically, the direction being vertical beneath the crests and troughs and horizontal beneath the nodes. (H.)

---

**Solution.** These are surface waves on still water, and hence satisfy conditions 6.2.1 (*d*)–(*i*)

$$\left.\begin{aligned} &\text{From } (d) \qquad \frac{\partial\phi}{\partial t} = ag\sin(2\pi x/\lambda)\cos(2\pi vt/\lambda) \\ &\text{From } (h) \qquad \frac{\partial\phi}{\partial y} = \frac{2\pi va}{\lambda}\sin(2\pi x/\lambda)\sin(2\pi vt/\lambda) \end{aligned}\right\}\text{ on } y = 0 \quad . \quad (1)$$

$$\text{From } (e) \qquad \frac{\partial\phi}{\partial y} = 0 \text{ on } y = -h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{From } (f) \qquad \nabla^2\phi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

By inspection of (1),

$$\phi = f(y)\sin(2\pi x/\lambda)\sin(2\pi vt/\lambda)$$

and substitution in (3) leads to

$$f'' - \left(\frac{2\pi}{\lambda}\right)^2 f = 0$$

i.e.
$$f(y) = Ae^{2\pi y/\lambda} + Be^{-2\pi y/\lambda}$$

while substitution of this in (2) gives

$$f(y) = C\cosh\left\{\frac{2\pi}{\lambda}(y+h)\right\}$$

Thus

$$\phi = C \cosh \left\{\frac{2\pi}{\lambda}(y+h)\right\} \sin(2\pi x/\lambda) \sin(2\pi\nu t/\lambda) \quad . \quad (4)$$

which gives, substituting in (1),

$$(2\pi\nu C/\lambda) \cosh 2\pi h/\lambda = ag$$

and

$$(2\pi C/\lambda) \sinh 2\pi h/\lambda = 2\pi\nu a/\lambda$$

i.e.

$$\left.\begin{aligned} \nu^2 &= \left(\frac{g\lambda}{2\pi}\right) \tanh(2\pi h/\lambda) \\ C &= a\nu/\sinh(2\pi h/\lambda) \end{aligned}\right\} \text{ as required}$$

and

For the path of any particle we use the velocity vector $\mathbf{q} = -\text{grad } \phi$. Using $\phi$ as above, $\mathbf{q}$ has components $q_1$, $q_2$ where

$$\left.\begin{aligned} q_1 &= -(2\pi C/\lambda) \cos(2\pi x/\lambda) \cosh\{2\pi(y+h)/\lambda\} \sin(2\pi\nu t/\lambda) \\ q_2 &= -(2\pi C/\lambda) \sin(2\pi x/\lambda) \sinh\{2\pi(y+h)/\lambda\} \sin(2\pi\nu t/\lambda) \end{aligned}\right\} \quad (5)$$

This gives the velocity components at any time $t$ of the water particle which is initially at $(x, y)$. From (5)

$$q_1/q_2 = \cot(2\pi x/\lambda) \coth\{2\pi(y+h)/\lambda\} \quad . \quad . \quad . \quad (6)$$

and so is independent of time; thus the *direction* of motion is unchanged, i.e. the particles all move in straight lines.

Also from (6) $q_1/q_2 = 0$, i.e. motion is vertical when $\cot(2\pi x/\lambda) = 0$, i.e. when $2\pi x/\lambda = (2n+1)\pi/2$; and similarly $q_1/q_2 \longrightarrow \infty$, i.e. motion is horizontal, when $2\pi x/\lambda = n\pi$.

But $2\pi x/\lambda = (2n+1)\pi/2$ gives $\sin(2\pi x/\lambda) = \pm 1$, i.e. gives the crests and troughs of the wave profile,

$$\eta = a \sin\left(\frac{2\pi x}{\lambda}\right) \cos\left(\frac{2\pi\nu t}{\lambda}\right)$$

and $2\pi x/\lambda = n\pi$ gives $\eta = 0$, i.e. gives the *nodes* of the wave profile.

Hence the particles move vertically beneath the crests and troughs and horizontally beneath the nodes.

Finally, to show that the particles move harmonically we consider, from (5), their total speed which is

$$q = (2\pi C/\lambda) \sin(2\pi\nu t/\lambda) f(x,y) \quad . \quad . \quad . \quad . \quad (7)$$

Thus a particle starting at $(x, y)$ will move, as above, in a straight line and its speed varies with time according to $\sin 2\pi vt/\lambda$; this is then simple harmonic motion with period $\lambda/v$, the period with which the whole profile moves.

---

**Comment.** The value given to $\lambda$ fixes the wavelength of the stationary waves; for stationary waves in a region between vertical planes, $\lambda$ may take only certain values, since then the condition $\dfrac{\partial\phi}{\partial x} = 0$ must hold on the bounding planes. The velocity $v$, determined by $\lambda$ and $h$ as above, then fixes the speed at which an individual particle, and the whole profile, move.

---

**Problem 80**

Establish the condition $\dfrac{\partial^2\phi}{\partial t^2} = g\dfrac{\partial\phi}{\partial y}$ at $y = 0$ on the velocity potential for small motions under gravity of an ideal incompressible fluid at the horizontal free surface $y = 0$, the $y$-axis being drawn downwards and surface tension being neglected.

Deduce the connection between wavelength and phase velocity (i.e. velocity of wave propagation) for plane progressive waves on deep water, making clear how the derivation depends on the condition that the water is deep.

Show that, if $z = x + iy$, $\phi =$ real part of $Az^{-1/2} \exp(-igt^2/4z)$ satisfies all the conditions of the problem, and sketch the shape of the surface for $x \geqslant \varepsilon > 0$, $t > 0$, $A$ being real. (L.)

---

**Solution.** The last part of the problem is in complex variable form, so it will be convenient to do the first part in this way (though it can, of course, be done otherwise, as in problem 75).

Using conditions 6.2.1, a plane progressive wave on water of depth $h$ may be represented by $\phi =$ real part of $A \cos(mz - nt - mih)$ since this makes $\nabla^2\phi = 0$ and $\dfrac{\partial\phi}{\partial y} = 0$ on $y = h$, the base ($y$ axis being now taken *downwards*).

The condition $\dfrac{\partial^2\phi}{\partial t^2} = g\dfrac{\partial\phi}{\partial y}$ at $y = 0$ then gives, as before,

$$n^2 = mg \tanh mh$$

or, if $c = n/m$, $\lambda = 2\pi/m$, $c^2 = \dfrac{\lambda g}{2\pi} \tanh\left(\dfrac{2\pi h}{\lambda}\right)$.

If the water is deep so that $2\pi h/\lambda$ is large, then $\tanh 2\pi h/\lambda = 1$ (in fact $1 > \tanh(2\pi h/\lambda) > 0{\cdot}99$ for $2\pi h/\lambda > 2{\cdot}65$). In this case

$$c^2 = \lambda g/2\pi$$

which is the required connection between wave velocity $c$ and wavelength $\lambda$.

If $\phi =$ real part of $Az^{-1/2}\exp(-igt^2/4z)$ then, on $y = 0$, we have

$$\frac{\partial^2\phi}{\partial t^2} = \text{real part of } Ax^{-1/2}\left(-g\frac{2t^2}{4x^2} - \frac{ig}{2x}\right)\exp(-igt^2/4x)$$

and $\dfrac{\partial\phi}{\partial y} =$ real part of $Ax^{-1/2}(-i/2x - gt^2/4x^2)\exp(-igt^2/4x)$

so that $\dfrac{\partial^2\phi}{\partial t^2} = \dfrac{\partial\phi}{\partial y}$ on $y = 0$ as required, i.e. the free surface condition is satisfied.

The stream function $\psi =$ imaginary part of

$$Az^{-1/2}\exp(-igt^2/4z)$$

and by inspection, $\psi \longrightarrow 0$ as $y \longrightarrow \infty$, all $x$ and $t$. Thus at a great depth the streamlines are parallel to the undisturbed free surface.

The given value of $\phi$ thus satisfies the conditions for surface waves on very deep water.

To find the shape of the free surface, use 6.2.1 (*d*)

$$\eta = \left(\frac{1}{g}\frac{\partial\phi}{\partial t}\right)_{y=0} = \text{real part of } -2Ait\,x^{-1/2}\exp(-igt^2/4x)$$

$$= \frac{2At}{\sqrt{x}}\sin\left(\frac{gt^2}{4x}\right)$$

This gives an elevation which is zero when $\dfrac{gt^2}{4x} = n\pi$, i.e. when

$$x = gt^2/4n\pi \text{ or } t^2 = 4\pi nx/g.$$

Thus:

(*a*) at constant time $t$ the surface is in the form of a modified sine wave which, as $x$ increases, has *decreasing* amplitude proportional to $x^{-1/2}$ and *increasing* wavelength;

(*b*) at constant distance $x$, the surface oscillates with *increasing* amplitude proportional to $t$, and *decreasing* period as $t$ increases;

i.e. for small $t = t_0$ the shape of the surface is shown by

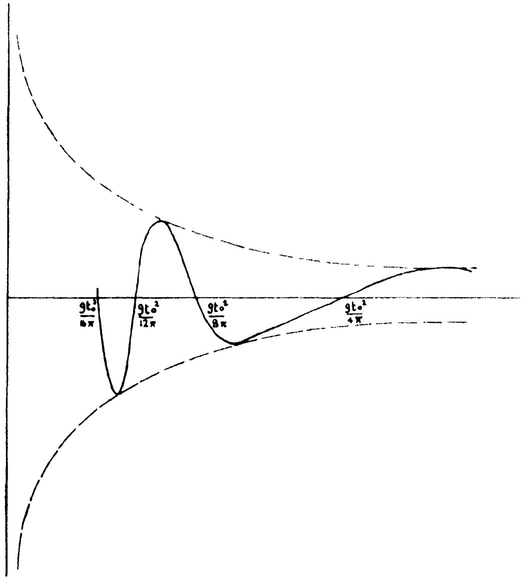


Fig. 53

for large $t = t_1$,

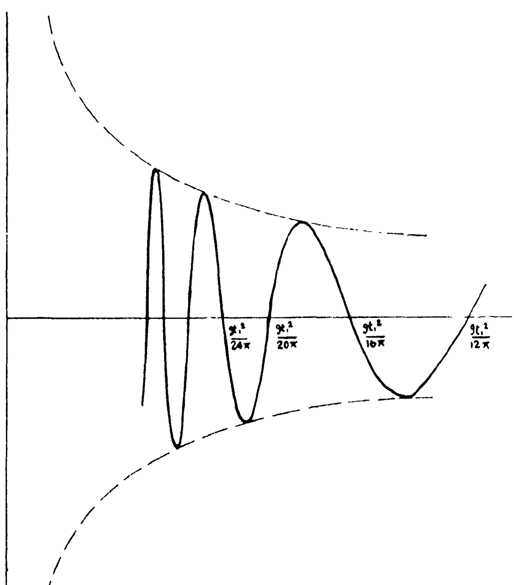


Fig. 54

**Problem 81**

Show that the equation for the propagation of tidal waves (long waves in shallow water) is

$$\frac{\partial}{\partial x}\left(h\frac{\partial\zeta}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial\zeta}{\partial y}\right)=\frac{1}{g}\frac{\partial^2\zeta}{\partial t^2}$$

where $\zeta$ is the height of the surface above the undisturbed water level and $h$ is the height of this level above the bottom of the tank.

If $h$ is given by $h = d + Kx$ ($d$, $K$, constant) prove that there are solutions of the form $\zeta = f(x)\cos pt$ where $p$ is a constant and find the equation satisfied by $f(x)$. Show that the solution is

$$\zeta = J_0(\alpha h^{1/2})\cos pt$$

where $\alpha$ is a constant and $J_0(\chi)$ satisfies the equation

$$\frac{d^2J_0}{d\chi^2}+\frac{1}{\chi}\frac{dJ_0}{d\chi}+J_0=0$$

and is finite at the origin.

If these standing waves are generated in a tank bounded by $x = 0$, $x = a$, $y = 0$, $y = b$, $z + d + Kx = 0$ ($a,b,d,K > 0$), find the additional restrictions imposed on the solution by the boundary conditions; find also the vertical velocity component of the water. (L.)

---

**Solution.** In dealing with long waves we use conditions (6.3):

$$(c)\quad\begin{cases} p = p_0 + g\rho(\zeta - z),\ p_0 \text{ being the pressure at } z = \zeta, \text{ the free surface} \\ \dfrac{\partial q_1}{\partial z}=0,\ \dfrac{\partial q_2}{\partial z}=0,\ \dfrac{\partial q_1}{\partial t}=-g\dfrac{\partial\zeta}{\partial x},\ \dfrac{\partial q_2}{\partial t}=-g\dfrac{\partial\zeta}{\partial y}\end{cases}$$

($d$) continuity, which for a tank of varying height $h$ gives, to the first order

$$\frac{\partial\zeta}{\partial t}+\frac{\partial(q_1h)}{\partial x}+\frac{\partial(q_2h)}{\partial y}=0\ . \quad . \quad . \quad . \quad . \quad (1)$$

by considering the flow of liquid into a small prism at $(x, y)$ of height $h + \zeta$, using the fact that the velocities $q_1$, $q_2$ are the same at all depths, and neglecting $q_1\zeta$, $q_2\zeta$ as second order quantities.

Differentiating (1) with respect to $t$ then gives

$$\frac{\partial^2\zeta}{\partial t^2}+h\frac{\partial}{\partial t}\left(\frac{\partial q_1}{\partial x}+\frac{\partial q_2}{\partial y}\right)+\frac{\partial}{\partial t}\left(q_1\frac{\partial h}{\partial x}+q_2\frac{\partial h}{\partial y}\right)=0$$

and, substituting from ($c$) for $\dfrac{\partial q_1}{\partial t}$, $\dfrac{\partial q_2}{\partial t}$ since $h$ is a function of $x$, $y$ only,

$$\frac{\partial^2\zeta}{\partial t^2} + h\frac{\partial}{\partial x}\left(-g\frac{\partial\zeta}{\partial x}\right) + h\frac{\partial}{\partial y}\left(-g\frac{\partial\zeta}{\partial y}\right) - g\frac{\partial\zeta}{\partial x}\frac{\partial h}{\partial x} - g\frac{\partial\zeta}{\partial y}\frac{\partial h}{\partial y} = 0$$

i.e. $$\frac{1}{g}\frac{\partial^2\zeta}{\partial t^2} = \frac{\partial}{\partial x}\left(h\frac{\partial\zeta}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial\zeta}{\partial y}\right) \text{ as required} \quad . \quad . \quad . \quad (2)$$

Substituting $h = d + Kx$ in (2) gives

$$(d + Kx)\frac{\partial^2\zeta}{\partial x^2} + K\frac{\partial\zeta}{\partial x} + (d + Kx)\frac{\partial^2\zeta}{\partial y^2} = \frac{1}{g}\frac{\partial^2\zeta}{\partial t^2}$$

which, if $\zeta = f(x)\cos pt$, reduces to

$$(d + Kx)f'' + Kf' + (p^2/g)f = 0 \quad . \quad . \quad . \quad (3)$$

which is the equation satisfied by $f$.

Putting $\chi = \alpha h^{1/2} = \alpha(d + Kx)^{1/2}$, equation (3) reduces to

$$\tfrac{1}{4}K^2\alpha^2\frac{d^2f}{d\chi^2} + \tfrac{1}{4}\chi K^2\alpha^2\frac{df}{d\chi} + \frac{p^2}{g}f = 0$$

i.e. to $$\frac{d^2f}{d\chi^2} + \frac{1}{\chi}\frac{df}{d\chi} + f = 0 \text{ if } \frac{4p^2}{gK^2\alpha^2} = 1 \quad . \quad . \quad . \quad . \quad (4)$$

This is Bessel's equation of zero order and has a solution $f = J_0(\chi)$, which is finite at the origin (this condition is necessary to maintain the basic assumption of small vertical displacement everywhere).

Since $K$ is fixed, the condition in (4) connects the parameter $\alpha$ with $p$, the wave frequency.

If waves of this form, where $\zeta = J_0(\alpha h^{1/2})\cos pt$, $h = d + Kx$, are to be generated in a tank bounded by $x = 0$, $x = a$, $y = 0$, $y = b$, $z + d + Kx = 0$ the conditions are simply that the normal velocity must vanish on each boundary.

The conditions specify a tank of rectangular section with a base sloping in the $x$-direction only, with depth varying from $d$ on $x = 0$ to $d + Ka$ on $x = a$, and so we require, from 6.3.1 ($e$),

$$\frac{\partial\zeta}{\partial x} = 0 \text{ on } x = 0, x = a \text{ and } \frac{\partial\zeta}{\partial y} = 0 \text{ on } y = 0, y = b$$

The value $h = 0$ is now excluded and so we may take

$$\zeta = \{AJ_0(\alpha h^{1/2}) + BY_0(\alpha h^{1/2})\}\cos pt$$

where $\tfrac{1}{2}\alpha Kh^{-1/2}\{AJ_0'(\alpha h^{1/2}) + BY_0'(\alpha h^{1/2})\}\cos pt = 0$ on $x = 0$, $x = a$

i.e. $$J_0'(\alpha d^{1/2})/Y_0'(\alpha d^{1/2}) = J_0'\{\alpha(d + Ka)^{1/2}\}/Y_0'\{\alpha(d + Ka)^{1/2}\} \quad (5)$$

This determines values of $\alpha$ ($d$, $K$, $a$ being given) and hence, from (4), determines possible wave frequencies $p$.

It should be noted that the condition of *long* waves—that the wavelength shall be long compared to the depth—implies the restriction that there must be only a *small* number of zeros of $J_0'$, $Y_0'$ between the ends, and this means that $Ka/d$ must be small, i.e. the total change in depth along the tank is small compared with the original depth.

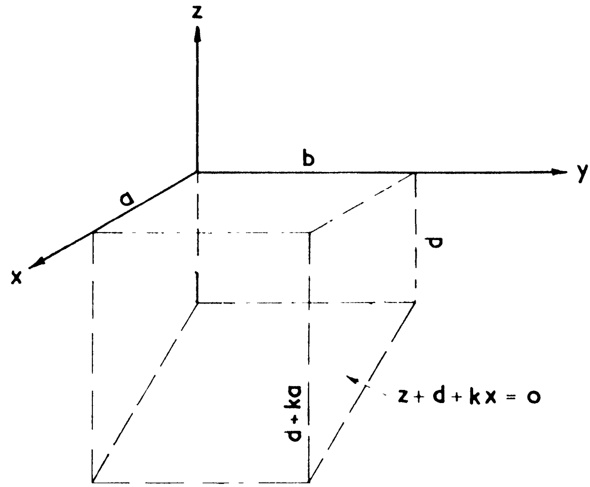


Fig. 55

To find the velocities, we use the original continuity equation (1), which now gives

$$\frac{\partial\zeta}{\partial t} + \frac{\partial(q_1 h)}{\partial x} = 0 \text{ or } h\frac{\partial q_1}{\partial x} + Kq_1 + \frac{\partial\zeta}{\partial t} = 0$$

As above $Ka/d$ is small and so $Kq_1$ is a second-order term compared with the others,

i.e. $$h\frac{\partial q_1}{\partial x} + \frac{\partial\zeta}{\partial t} = 0 \text{ to the first order}$$

By the general continuity equation 6.1 (*a*), div $\mathbf{q} = 0$

i.e. $\dfrac{\partial q_1}{\partial x} + \dfrac{\partial q_3}{\partial z} = 0$ and so $\dfrac{\partial q_3}{\partial z} = \dfrac{1}{h}\dfrac{\partial\zeta}{\partial t}$ (6) and $q_3 = 0$ on $z = -h$ (since the slope of the base is small)

Substituting for $\zeta$ and integrating gives

$$q_3 = -\frac{p(z+h)}{h}\zeta \tan pt$$

for the vertical velocity.

It should be noted that condition (6) is exactly the same as would be obtained by equating the vertical velocity of the profile $\frac{\partial \zeta}{\partial t}$ to the particle speed at the surface. This follows since $\frac{\partial q_3}{\partial z}$ is independent of $z$ and so, since $q_3 = 0$ on $z = -h$, the surface speed is $h\, \partial q_3/\partial z$.

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**Comment.** The assumption of long waves implies restrictions on the geometrical conditions—in particular, that if there are changes in breadth or depth the amount of change shall be small within a wavelength.

## PROBLEMS FOR SOLUTION

1. An inviscid liquid of constant density $\rho$ occupies the region $-\infty < x < \infty$, $a \sin (2\pi/\lambda)(x - ct) < y < \infty$, where $Oy$ is drawn in the vertically downward direction. $y = a \sin (2\pi/\lambda)(x - ct)$ is the free surface of the liquid at time $t$, and $a$ is infinitesimally small compared with $\lambda$. Verify that the velocity potential $\phi$ for the consequent irrotational wave motion is given by

$$\phi = ac \exp (-2\pi y/\lambda) \cos (2\pi/\lambda)(x - ct)$$

where $c^2 = g\lambda/2\pi$. Show that the total energy per wavelength is $\frac{1}{2}g\rho\lambda a^2$.

If the group velocity $c_g$ is defined by $c_g = c - \lambda\frac{dc}{d\lambda}$, show that the energy is transmitted with the group velocity. (L.)

2. A stream of depth $h$ flowing with uniform horizontal velocity $U$ in the $x$ direction is slightly disturbed by irregularities $a \sin Kx$ in the bed of the stream, $a$ being small compared with $2\pi/K$ and with $h$. Show that the first-order stream function for the steady disturbance is

$$\psi = \frac{U \sin Kx}{\sinh Kh}\{a \sinh K(y - h) - b \sinh Ky\}$$

where $y$ is measured vertically upwards from the bed and $b$ represents the amplitude of the disturbance of the free surface. Find the ratio $b/a$ and show that the elevation of the free surface is either in phase or in antiphase with the elevation of the bed according as $U^2 \lessgtr \left(\frac{g}{K}\right) \tanh Kh$. What happens when $U^2 = \left(\frac{g}{K}\right) \tanh Kh$ and why? (L.)

3. An infinite liquid of density $\rho'$ lies above an infinite liquid of density $\rho$ the surface of separation being horizontal. Allowing for surface tension $T$, show that the velocity $c$ of propagation of waves of length $\lambda$ is given by

$$c^2 = g\frac{\lambda}{2\pi}\frac{(\rho - \rho')}{(\rho + \rho')} + \frac{2\pi T}{\lambda(\rho + \rho')}$$

If $c^2$ has a minimum value when $\lambda = \lambda_0$, show that the group velocity is

$$c\left\{1 - \frac{(\lambda^2 - \lambda_0^2)}{2(\lambda^2 + \lambda_0^2)}\right\} \qquad \text{(D.)}$$

4. A deep liquid of density $\rho'$ is moving with speed $V'$ over another deep liquid of density $\rho$ moving in the same direction with speed $V$. If $c$ is the velocity of propagation of oscillatory waves of length $2\pi/m$ at the interface, show that

$$(V - c)^2\rho + (V' - c)^2\rho' = g(\rho - \rho')/m$$

If the velocity of wind is just sufficient to prevent propagation of waves against it on deep water at rest but for the wave motion, show that the velocity of propagation of waves with the wind is $\dfrac{2c\sqrt{\sigma(1-\sigma)}}{1+\sigma}$ where $\sigma$ is the specific gravity of air and $c$ is the propagated wave velocity if there were no air present. (N.)

5. Obtain the equation of propagation of long waves in a channel of cross-section of varying area $S$ in the form

$$b\frac{\partial^2\eta}{\partial t^2} = g\frac{\partial}{\partial x}\left(S\frac{\partial\eta}{\partial x}\right)$$

where $x$ is measured along the channel, $b$ is the breadth at the free surface, and $\eta$ is the height of the free surface above the mean level.

Show that if the cross-section changes abruptly at any point, $\eta$ and $S\partial\eta/\partial x$ are continuous.

Hence prove that if the section changes abruptly from a rectangle of breadth $a$ to one of breadth $b$, the depths on either side of the discontinuity being the same, an incident wave of amplitude $\eta_0$ in the channel of breadth $a$ gives rise to a reflected wave of amplitude $\eta_0(a - b)/(a + b)$ and a transmitted wave of amplitude $2\eta_0 a/(a + b)$. (L.)

6. Waves of small elevation $a \sin (mx - nt)$ are moving on water of depth $h$. Show that, with proper choice of axes, the complex potential of the motion is $w = ac \operatorname{cosech} mh \cos (mz - nt)$, where $c^2 = (g/m) \tanh mh$. Examine the variation of pressure at any given point on the bed as the waves pass along. (L.)

7. A harmonic train of progressive waves of small amplitude $a$ and wave length $\lambda$ moves under gravity upon the surface of water of infinite extent and uniform depth $h$. Show that the velocity $v$ of the waves is given by $v^2 = \dfrac{g\lambda}{2\pi} \tanh (2\pi h/\lambda)$.

Show also that during the motion each particle of water describes an ellipse, the distance between the foci of which is $2a \operatorname{cosech} (2\pi h/\lambda)$. (H.)

8. A group of straight-crested waves of wavelengths all very close to $2\pi/K$ are propagated with velocity $V$. Show that the maxima of the resulting wave profile travel with velocity $U$, where $U = d(KV)/dK$.

If the waves are gravity waves in water of constant depth $h$, find $V$ and $U$ and show that $\frac{1}{2} < U/V < 1$.

Show that in this case $U$ is also the average rate at which energy is transmitted across a fixed plane normal to the direction of propagation.

9. If a horizontal rectangular canal of great depth has two vertical barriers at a distance $l$ apart, prove that the periods of possible stationary surface waves are $2\sqrt{\pi l}/\sqrt{sg}$ where $s$ is a positive integer; and that, for any value of $s$, the motion is such that all the particles of fluid oscillate in straight lines of length inversely proportional to $\exp (s\pi z/l)$, where $z$ is the depth below the undisturbed free surface.

10. Show that the velocity potential for simple harmonic waves of small amplitude $a$ and elevation $\eta = a \sin \{2\pi(vt - x)/\lambda\}$ on the surface of water of infinite extent and uniform depth $h$ is

$$\phi = -av \cos \{2\pi(vt - x)/\lambda\} \cosh \{2\pi(y + h)/\lambda\} \operatorname{cosech} 2\pi h/\lambda$$

and hence or otherwise find the velocity of propagation in terms of $\lambda$ and $h$.

Show also that the pressure at height $\frac{1}{2}h$ above the bottom is

$$g\rho\{\tfrac{1}{2}h + \eta \cosh (\pi h/\lambda) \operatorname{sech} (2\pi h/\lambda)\}$$

where $\rho$ is the density. (H.)

11. Find the possible lengths of standing waves in a rectangular trough of length $2l$ containing water to a uniform depth $h$, surface tension being neglected.

The undisturbed surface of the water is in the plane $y = 0$, the base is in the plane $y = -h$, and the ends of the trough are in the planes $x = \pm l$. The water is displaced so that initially it is at rest, and the form of its surface is given by $\eta = a \sin(\pi x/2l)$, where $a$ is small. Find the form of the surface at any subsequent time $t$: (i) when surface tension is neglected; (ii) when surface tension is taken into account. (L.)

# CHAPTER 7
# VISCOUS FLOW

## 7.1 Basic Equations—Navier–Stokes Equations

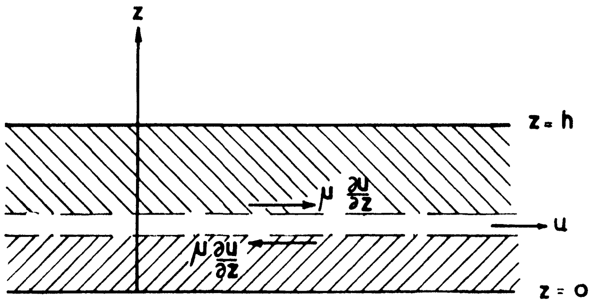


Fig. 56

To define the coefficient of viscosity $\mu$ consider a fluid in horizontal motion with velocity $u$ between two parallel horizontal planes $z = 0$, $z = h$; then the two portions of the fluid separated by a horizontal plane exert on each other a tractive force $\mu \frac{\partial u}{\partial z}$ per unit area. This definition can be generalised to Stokes' Law, which states that viscous stresses are proportional to the rate of change of strain with time.

Then for incompressible liquid of constant viscosity the basic equations governing the flow are those for non-viscous liquid (3.1) with additional terms arising from this frictional stress. We have:

an equation of continuity

$$\operatorname{div} \mathbf{q} = 0$$

an expression for acceleration

$$\mathbf{f} = \left[\frac{\partial \mathbf{q}}{\partial t}\right] + (\mathbf{q}.\nabla\mathbf{q})$$

as in (3.1)

(where $\left[\frac{\partial \mathbf{q}}{\partial t}\right] = \frac{\partial \mathbf{q}}{\partial t} + \Omega \times \mathbf{q}$ if the axes rotate with angular velocity $\Omega$)

* and a basic equation of motion which can be shown to be

$$\mathbf{f} = \mathbf{F} - \frac{1}{\rho} \operatorname{grad} p + \nu\nabla^2\mathbf{q} \quad . \quad . \quad . \quad . \quad (1)$$

* The notation $\mathbf{f} = \frac{D\mathbf{q}}{Dt}$ is used in many textbooks, where $\frac{D}{Dt}$ denotes the operator $\frac{\partial}{\partial t} + (\mathbf{q}\,.\,\nabla) + \Omega\,\times$

where $\nu = \mu/\rho$, $\mu$ being the coefficient of viscosity and $\rho$ the density. ($\nu$ is called the kinematic viscosity).

The equations (1) are known as the Navier–Stokes equations.

**7.1.1 Stresses in the Liquid.** There are nine components of stress, and a double suffix notation is used to describe them; thus $p_{ij}$ is the stress (force per unit area) *on* a plane normal to the axis **i** *in the direction* of the axis **j**.

The normal stress $p_{ii}$ is taken as positive when it is a *tension*, negative when a *compression*. With this notation and convention, the stress components may be expressed in terms of velocity gradients: in the coordinate system $(x, y, z)$ the relations are

$$p_{xx} = -p + 2\mu \frac{\partial q_1}{\partial x}$$

$$p_{yz} = \mu\left(\frac{\partial q_3}{\partial y} + \frac{\partial q_2}{\partial z}\right) = p_{zy}$$

with similar relations for $p_{yy}, p_{zz}, p_{zx}, p_{xy}$. Here $p$ is the mean pressure at a point.

**7.1.2 Equations of Motion and Stress Functions in other Coordinate Systems.** The basic equations may be expressed in any coordinate system by transforming the velocity components and using the relations given in 1.1.5. It is convenient, however, to write down explicitly the equations of continuity and motion and the stresses in the two most common other coordinate systems, cylindricals and spherical polars.

(*a*) *Cylindrical Coordinates* $(r, \theta, z)$

*Continuity* $$\frac{\partial q_r}{\partial r} + \frac{q_r}{r} + \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{\partial q_z}{\partial z} = 0$$

*Equations of Motion*

$$\frac{Dq_r}{Dt} - \frac{q_\theta^2}{r} = F_r - \frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 q_r - \frac{2}{r^2}\frac{\partial q_\theta}{\partial \theta} - \frac{q_r}{r^2}\right)$$

$$\frac{Dq_\theta}{Dt} + \frac{q_r q_\theta}{r} = F_\theta - \frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 q_\theta + \frac{2}{r^2}\frac{\partial q_r}{\partial \theta} - \frac{q_\theta}{r^2}\right)$$

$$\frac{Dq_z}{Dt} = F_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\,\nabla^2 q_z$$

where $$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + q_r\frac{\partial}{\partial r} + q_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + q_z\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + (\mathbf{q}.\nabla)$$

and $$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{dr} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

and the other terms on the left-hand side are components of $\Omega \times \mathbf{q}$

*Stresses*

$$p_{rr} = -p + 2\mu\frac{\partial q_r}{\partial r}; p_{\theta\theta} = -p + 2\mu\left(\frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + q_r/r\right);$$

$$p_{zz} = -p + 2\mu\frac{\partial q_z}{\partial z}; p_{\theta z} = \mu\left(\frac{1}{r}\frac{\partial q_z}{\partial \theta} + \frac{\partial q_\theta}{\partial z}\right);$$

$$p_{zr} = \mu\left(\frac{\partial q_r}{\partial z} + \frac{\partial q_z}{\partial r}\right); p_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial q_r}{\partial \theta} - q_\theta/r + \frac{\partial q_\theta}{\partial r}\right)$$

(*b*) *Spherical Polar Coordinates* $(r, \theta, \tilde{\omega})$

*Continuity* $$\frac{\partial q_r}{\partial r} + \frac{2q_r}{r} + \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{q_\theta \cot\theta}{r} + \frac{1}{r\sin\theta}\frac{\partial q_{\tilde{\omega}}}{\partial \tilde{\omega}} = 0$$

*Equations of Motion*

$$\frac{Dq_r}{Dt} - \frac{q_\theta^2 + q_{\tilde{\omega}}^2}{r} = F_r - \frac{1}{\rho}\frac{\partial p}{\partial r}$$
$$+ \nu\left(\nabla^2 q_r - \frac{2q_r}{r^2} - \frac{2\cot\theta}{r^2}q_\theta - \frac{2}{r^2}\frac{\partial q_\theta}{\partial \theta} - \frac{2}{r^2\sin\theta}\frac{\partial q_{\tilde{\omega}}}{\partial \tilde{\omega}}\right)$$

$$\frac{Dq_\theta}{Dt} - \frac{q_{\tilde{\omega}}^2\cot\theta}{r} + \frac{q_r q_\theta}{r} = F_\theta - \frac{1}{\rho r}\frac{\partial p}{\partial \theta}$$
$$+ \nu\left(\nabla^2 q_\theta - \frac{q_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial q_r}{d\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial q_{\tilde{\omega}}}{\partial \tilde{\omega}}\right)$$

$$\frac{Dq_{\tilde{\omega}}}{Dt} + \frac{q_r q_{\tilde{\omega}}}{r} + \frac{q_\theta q_{\tilde{\omega}}\cot\theta}{r} = F_{\tilde{\omega}} - \frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial \tilde{\omega}}$$
$$+ \nu\left(\nabla^2 q_{\tilde{\omega}} - \frac{q_{\tilde{\omega}}}{r^2\sin^2\theta} + \frac{2}{r^2\sin^2\theta}\frac{\partial q_r}{\partial \tilde{\omega}} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial q_\theta}{\partial \tilde{\omega}}\right)$$

where $$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + q_r\frac{\partial}{\partial r} + q_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + q_{\tilde{\omega}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \tilde{\omega}} = \frac{\partial}{\partial t} + (\mathbf{q}.\nabla)$$

and $$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \tilde{\omega}^2}$$

and the other terms on the left-hand side are components of $\Omega \times \mathbf{q}$

*Stresses*

$$p_{rr} = -p + 2\mu\frac{\partial q_r}{\partial r}; p_{\theta\theta} = -p + 2\mu\left(\frac{\partial q_\theta}{r\partial \theta} + \frac{q_r}{r}\right);$$

$$p_{\tilde{\omega}\tilde{\omega}} = -p + 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial q_{\tilde{\omega}}}{\partial \tilde{\omega}} + \frac{q_r}{r} + \frac{q_\theta\cot\theta}{r}\right)$$

$$p_{\theta\tilde{\omega}} = \mu\left(\frac{1}{r}\frac{\partial q_{\tilde{\omega}}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial q_\theta}{\partial \tilde{\omega}} - \frac{q_{\tilde{\omega}}\cot\theta}{r}\right);$$

$$p_{\tilde{\omega}r} = \mu\left(\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial \tilde{\omega}} - \frac{q_{\tilde{\omega}}}{r} + \frac{\partial q_{\tilde{\omega}}}{\partial r}\right)$$

$$p_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial q_r}{\partial \theta} - \frac{q_\theta}{r} + \frac{\partial q_\theta}{\partial r}\right)$$

**7.1.3 Boundary Conditions.** A viscous fluid does not slip in contact with a boundary; i.e. *both* the tangential and normal velocity components of fluid and boundary are the same.

## 7.2 Dynamical Similarity, Reynolds' Number

A necessary condition for the dynamical similarity of two fluid systems which are geometrically similar is that they have the same Reynolds' number, $R$, defined by

$$R = ul/\nu$$

where $u$, $l$ are respectively standard velocity and standard length and $\nu = \mu/\rho$ as above.

$$\left.\begin{array}{ll}\text{Since } u \text{ has dimensions} & LT^{-1}\\ \quad l \text{ has dimensions} & L\\ \quad \mu \text{ has dimensions} & ML^{-1}T^{-1}\\ \quad \rho \text{ has dimensions} & ML^{-3}\end{array}\right\} R \text{ is non-dimensional.}$$

If $R$ is small viscosity effects predominate.
If $R$ is large inertia effects predominate.

## 7.3 Approximate Solutions

Few exact solutions of the equations 7.1 (1) are known, and it has been found necessary to make approximations. These are

(*a*) *Small Reynolds' Number—Slow or Creeping Motion.* As a first approximation this case can be considered as covered by the equations 7.1 (1) with the inertia terms omitted, i.e. by

$$\frac{1}{\rho}\operatorname{grad} p = \nu\nabla^2\mathbf{q} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

together with continuity and boundary conditions as before.

Extensions of this formula taking the inertia terms partly into account have been used, e.g. by Oseen in considering the drag on a sphere.

(*b*) *Large Reynolds' Number—Boundary Layer Theory.* This case can be dealt with by Prandtl's assumption of a boundary layer; the effects of viscosity are supposed to be perceptible only within a layer of fluid close to the boundary, and elsewhere the flow is to be non-viscous. The derivation of the corresponding equations is given in the solution to Problem 90.

In the problems which follow, numbers 81–84 deal with exact solutions of the Navier–Stokes equations; number 85 is a solution using the method of dimensions only; numbers 86–89 deal with the approximation of very slow motion; numbers 90, 91 give the derivations of the Prandtl boundary layer equations and von Karman's momentum integral equation and solve two simple problems.

**Problem 82**

The space above the plane $y = 0$ is filled with liquid of kinematic viscosity $\nu$. Initially the plane and liquid are at rest. At time $t = 0$ the plane begins to move parallel to itself with velocity $U$ and the liquid also moves in the same direction. Show that the equation of motion is

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

and that the appropriate solution is

$$u = U - \frac{2U}{\sqrt{\pi}} \int_0^{\alpha} e^{-t^2}\, dt$$

where $\alpha = y/(4\nu t)^{1/2}$ (O.)

**Solution.**

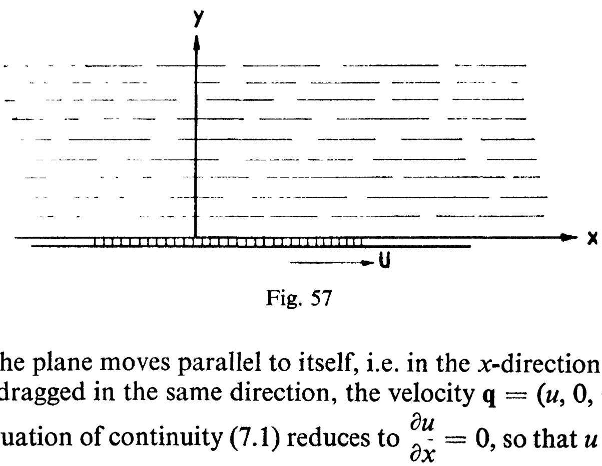


Fig. 57

Since the plane moves parallel to itself, i.e. in the $x$-direction, and the liquid is dragged in the same direction, the velocity $\mathbf{q} = (u, 0, 0)$.

The equation of continuity (7.1) reduces to $\frac{\partial u}{\partial x} = 0$, so that $u$ does not depend on $x$. Further, since the motion is generated by the motion of an infinite plane, it must be two-dimensional, i.e. all quantities must be independent of $z$.

Hence $u = f(y, t)$ and $u = U$ when $y = 0$, $u \longrightarrow 0$ as $y \longrightarrow \infty$ (i.e. the fluid has the same velocity as the moving boundary, and remains at rest at a great distance from this boundary).

Substituting in the first of the Navier–Stokes equations (7.1) gives

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and this means that $\frac{\partial p}{\partial x}$ must be independent of $x$, since $u$ is. The other equations reduce simply to $\frac{\partial p}{\partial y} = 0$, $\frac{\partial p}{\partial z} = 0$, so that $p$ is independent of

$y$ and $z$. Further, since the plane is infinite in the $x$-direction, there cannot in fact be a pressure gradient in this direction, so that $\dfrac{\partial p}{\partial x} = 0$. Thus $p$ can depend only on time, and the equation (1) reduces to

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \quad \ldots \ldots \quad (2)$$

This can be transformed to an ordinary differential equation by the substitutions

$$\alpha = y/(4\nu t)^{1/2}, \ u = g(\alpha)$$

since then $$\frac{\partial u}{\partial t} = -\frac{\alpha}{2t}\frac{du}{d\alpha}, \ \frac{\partial^2 u}{\partial y^2} = \frac{1}{4\nu t}\frac{d^2 u}{d\alpha^2}$$

(2) then becomes $$-2\alpha \frac{du}{d\alpha} = \frac{d^2 u}{d\alpha^2} \quad \ldots \ldots \quad (3)$$

and the boundary conditions are $u = U$, $\alpha = 0$, and $u \longrightarrow 0$ as $\alpha \longrightarrow \infty$.

The general solution of (3) is

$$u = A\int_0^\alpha e^{-t^2}\,dt + B$$

and the boundary conditions give

$B = U$, $A = -U/\int_0^\infty e^{-t^2}\,dt = -2U/\sqrt{\pi}$ using the known integral.

Thus $u = U - \dfrac{2U}{\sqrt{\pi}}\displaystyle\int_0^\alpha e^{-t^2}\,dt$ as required.

**Comment.** The results that $u$ is a function of $y$, $t$ only, and $p$ is a function of $t$ only, could have been written down intuitively on consideration of the geometry of the problem; the full discussion has been given here to illustrate the way in which, without appealing to intuition, the results may be found by straightforward application of the basic equations.

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## Problem 83

Assuming the equation of motion of viscous liquid to be

$$\frac{D\mathbf{q}}{Dt} = \rho\mathbf{F} - \nabla p + \mu\nabla^2\mathbf{q}$$

show that if liquid flows steadily under pressure only through a long cylindrical pipe of radius $a$ with a coaxial core of radius $b$, the velocity at a distance $r$ from the axis is of the form $A + B\log r + \frac{1}{4}Cr^2/\mu$.

If end effects may be neglected, determine the constants given that the

pipe is of length $l$ and that the pressures at the ends are $p$, $p^1$ ($p > p^1$). Find the mean speed of flow and the drag per unit area on the wall of the pipe. (L.)

---

**Solution.** Using cylindrical coordinates $(r, \theta, z)$ with the $z$-axis along the pipe, we have steady flow in the $z$-direction only, so that

$$\mathbf{q} = (0, 0, v)$$

The equation of continuity (7.1) reduces to $\frac{\partial v}{\partial z} = 0$, so $v$ must depend on $r$ only.

The Navier–Stokes equations (7.1 (1) using 7.1.2 ($a$)) reduce, since $\mathbf{F} = 0$, to

$$\frac{\partial p}{\partial r} = 0; \frac{1}{r}\frac{\partial p}{\partial \theta} = 0 \quad \quad (1)$$

and

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr}\right) \quad \quad (2)$$

From (1), $p$ does not depend on $r$ or $\theta$ and, from (2), $\frac{\partial p}{\partial z}$ does not depend on $z$, since $v$ does not. Thus $\frac{\partial p}{\partial z}$ must be *constant* along the pipe; let the constant be $C$. Then (2) is

$$\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} = \frac{C}{\nu\rho} = \frac{C}{\mu} \quad \quad (3)$$

and the solution of this is

$$v = A + B\log r + \tfrac{1}{4}Cr^2/\mu \text{ as required} \quad \quad (4)$$

If the pipe is of length $l$, the pressures at the ends are $p$, $p^1$ and end effects are neglected, then $C = \frac{\partial p}{\partial z} = -\left(\frac{p - p^1}{l}\right)$; the pressure gradient must be negative to produce a positive flow in the $z$-direction.

The other conditions are fixed by the boundary conditions $v = 0$ on $r = a$, $v = 0$ on $r = b$. Substituting these in (4) gives

$$\left.\begin{aligned} A &= \left(\frac{p - p^1}{4\mu l}\right)(b^2 \log a - a^2 \log b)/\log(a/b) \\ B &= \left(\frac{p - p^1}{4\mu l}\right)(a^2 - b^2)/\log(a/b) \end{aligned}\right\} \quad \quad (5)$$

The mean speed of flow is $\frac{1}{\pi(a^2 - b^2)}\int_b^a 2\pi r v\, dr$ and substituting from (4) and (5) and integrating gives finally

$$\text{mean speed} = \left(\frac{p - p^1}{4\mu l}\right)\{(a^2 + b^2)\log(a/b) - (a^2 - b^2)\}/2\log(a/b)$$

The drag on the wall of the pipe per unit area is $p_{rz}$ (drag on a surface $r$ = constant in the $z$-direction) and, from 7.1.2 ($a$), this is

$$p_{rz} = \mu\left(\frac{dv}{dr}\right)_{r=a}$$

which, substituting from (4) and (5) and differentiating, gives

$$\text{drag per unit area} = \left(\frac{p - p^1}{4l}\right)\{a^2 - b^2 - 2a^2 \log (a/b)\}/a \log (a/b)$$

**Problem 84**

The velocity at any point $P(x, y, z)$ in a viscous incompressible fluid is $\omega r$ perpendicular to the plane through $P$ and the $z$-axis, where $r^2 = x^2 + y^2$ and $\omega$ is a function of $r$ and $t$ only. Prove that the tangential stress at any point on the cylinder $r$ = constant is $\mu r \dfrac{\partial \omega}{\partial r}$.

Prove also that if the external forces have no moment about the $z$-axis,

$$\frac{\partial^2 \omega}{\partial r^2} + \frac{3}{r}\frac{\partial \omega}{\partial r} = \frac{1}{\nu}\frac{\partial \omega}{\partial t}$$

In the case when the motion is steady and takes place between two rigid cylindrical boundaries $r = a$, $r = b$ ($b > a$), which are rotating about their axes with constant angular velocities $\omega_1$, $\omega_2$, respectively, show that

$$\omega = \frac{1}{b^2 - a^2}\left\{\frac{a^2b^2(\omega_1 - \omega_2)}{r^2} + \omega_2 b^2 - \omega_1 a^2\right\}$$

and that the external couple on unit length of the inner cylinder necessary to maintain the motion is

$$4\pi\mu a^2b^2(\omega_1 - \omega_2)/(b^2 - a^2) \qquad \text{(L.)}$$

**Solution.**

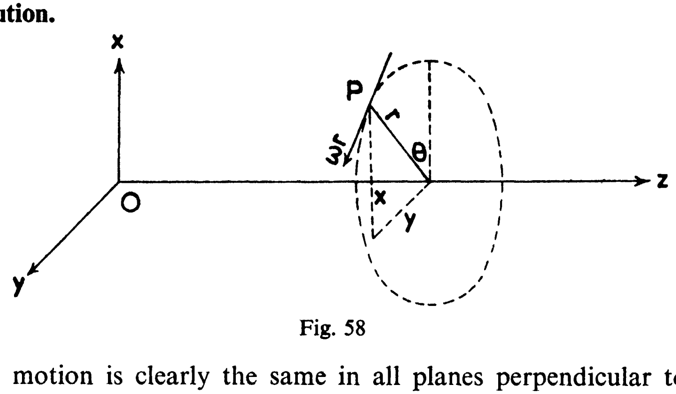


Fig. 58

The motion is clearly the same in all planes perpendicular to the $z$-axis, and so none of the quantities can depend on $z$. Taking cylindrical coordinates $r$, $\theta$, $z$, the tangential stress on the cylinder $r$ = constant

(i.e. stress in the direction of $\theta$ increasing) is $p_{r\theta}$, which, from 7.1.2 (*a*), is

$$\mu\left(\frac{1}{r}\frac{\partial q_r}{\partial\theta} - \frac{q_\theta}{r} + \frac{\partial q_\theta}{\partial r}\right)$$

Here we are given that $q_r = 0, q_\theta = \omega r$ and $\omega$ depends on $r$ and $t$ only,

so that
$$p_{r\theta} = \mu r\frac{\partial\omega}{\partial r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the external forces have no moment about the $z$-axis they must be wholly in the $r$-direction (there can be no force in the $z$-direction).

The basic equations 7.1.2 (*a*) therefore reduce to

$$-\omega^2 r = F_r - \frac{1}{\rho}\frac{\partial p}{\partial r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$r\frac{\partial\omega}{\partial t} = \nu\left(3\frac{\partial\omega}{\partial r} + r\frac{\partial^2\omega}{\partial r^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and (3) gives $\dfrac{\partial^2\omega}{\partial r^2} + \dfrac{3}{r}\dfrac{\partial\omega}{\partial r} = \dfrac{1}{\nu}\dfrac{\partial\omega}{dt}$ as required.

If the motion is *steady*, $\dfrac{\partial\omega}{\partial t} = 0$, and (3) becomes

$$\frac{d^2\omega}{dr^2} + \frac{3}{r}\frac{d\omega}{dr} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

(an ordinary differential equation now, since $\omega$ depends on $r$ only).

The general solution of this is $\omega = A/r^2 + B$ and the solution satisfying the given boundary conditions $\omega = \omega_1$, $r = a$; $\omega = \omega_2$, $r = b$ is

$$\omega = \frac{1}{b^2 - a^2}\left\{\frac{a^2b^2(\omega_1 - \omega_2)}{r^2} + \omega_2 b^2 - \omega_1 a^2\right\} \text{ as required}$$

Since $p_{r\theta} = \mu r\dfrac{d\omega}{dr}$, its value on the inner cylinder is

$$p_{a\theta} = -\frac{2\mu b^2(\omega_1 - \omega_2)}{b^2 - a^2}$$

and the frictional couple on unit length of the inner cylinder is

$$2\pi a^2 p_{a\theta}$$

i.e.
$$-4\pi\mu a^2b^2(\omega_1 - \omega_2)/(b^2 - a^2)$$

The external couple necessary to maintain the motion is then equal and opposite to this.

---

**Problem 85**

Liquid of viscosity $\mu$ is bounded by $n$ rigid cylindrical surfaces $C_n$ with generators parallel to the $z$-axis, which move with velocities $V_n$ in

the direction of their lengths. If the liquid motion is steady with velocity $q$, everywhere parallel to the $z$-axis, and $p$ is the pressure, show that

(i) $q = q(x, y)$ (ii) $\dfrac{dp}{dz} = -k$ (constant) (iii) $4\mu q = 2\phi - kr^2$

where $r = (x^2 + y^2)^{1/2}$ and $\phi$ is a plane harmonic function satisfying

$$\phi = \tfrac{1}{2}kr^2 + 2\mu V_n$$

on each boundary $C_n$.

Determine the general form of $\phi$ when $\phi = \phi(r)$ only.

If the liquid is contained between two boundaries $C_1$, $C_2$ which are coaxial circular cylinders of cross-sectional areas $S_1$, $S_2$, respectively ($S_1 < S_2$), show that if $k = 0$ the flux past any cross-section is

$$S_2V_2 - S_1V_1 + \frac{(S_2 - S_1)(V_1 - V_2)}{\log (S_2/S_1)} \qquad \text{(L.)}$$

**Solution.**

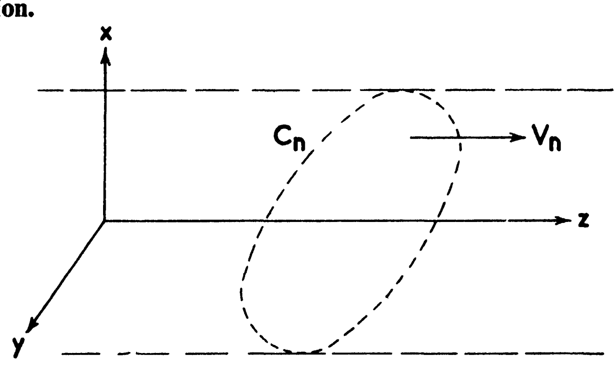


Fig. 59

The motion is to be steady with velocity $\mathbf{q}$ everywhere parallel to the $z$-axis, i.e $\mathbf{q} = (0, 0, q)$.

The basic equations (7.1) reduce to

$$\frac{\partial p}{\partial x} = 0, \frac{\partial p}{\partial y} = 0 \qquad (1)$$

and

$$q\frac{\partial q}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 q \qquad (2)$$

and the equation of continuity is $\dfrac{\partial q}{\partial z} = 0$.

Thus:

(i) $q = q(x, y)$

(ii) $p$ is a function of $z$ only, but $\dfrac{\partial p}{\partial z}$, from (2), cannot depend on $z$,

since $q$ does not; hence $\dfrac{\partial p}{\partial z} = -k$, a constant (negative to produce a positive velocity $q$)

(iii) substituting from (i) and (ii) in (2) gives

$$-k/\rho = \nu\left(\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2}\right)$$

i.e.
$$-k/\mu = \frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let $\phi = \phi(x, y)$ be any plane harmonic, i.e. a solution of $\nabla^2\phi = 0$. Then a general solution of (3) is

$$q = \phi/2\mu - k(x^2 + y^2)/4\mu \text{ as required} \quad . \quad . \quad . \quad (4)$$

The boundary conditions are $q = V_n$ on $C_n$, i.e., substituting in (4),

$$\phi = 2\mu V_n + \tfrac{1}{2}kr^2 \text{ as required} \quad . \quad . \quad . \quad . \quad (5)$$

If $\phi = \phi(r)$ only, then it satisfies (see 7.1.2 (*a*))

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} = 0$$

and so is of general form

$$\phi = A \log r + B \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If the liquid is contained between coaxial cylinders, $C_1$, $C_2$, radii $r_1$, $r_2$, and $k = 0$, then (5) and (6) must both hold on both $C_1$ and $C_2$, and we have

$$\left.\begin{aligned} A &= 2\mu(V_2 - V_1)/\log(r_2/r_1) \\ B &= 2\mu(V_1 \log r_2 - V_2 \log r_1)/\log(r_2/r_1) \end{aligned}\right\} \quad . \quad . \quad (7)$$

The flux past any cross-section is $\int 2\pi rq\, dr$, and, from (4) with $k = 0$,

$$q = \phi/2\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Substituting the value for $\phi$ from (6) and (7) and integrating gives for the total flux

$$\pi(r_2^2 V_2 - r_1^2 V_1) + \frac{(V_1 - V_2)\,\pi\,(r_2^2 - r_1^2)}{2 \log(r_2/r_1)} \quad .$$

or $S_2V_2 - S_1V_1 + (V_1 - V_2)(S_2 - S_1)/\log(S_2/S_1)$ as required.

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**Problem 86**

Use dimensional considerations to determine, apart from a numerical factor, the drag exerted on a small sphere moving with constant speed through a viscous fluid.

Deduce the rate of rise of a small spherical air bubble through a viscous liquid in terms of its radius. If bubbles are released at different depths with the same initial radius, show that the time taken for a bubble to reach the free surface of the liquid is proportional to $p_1 - p_0{}^{5/3}/p_1{}^{2/3}$ where $p_0$, $p_1$ are the pressures at the free surface and at the point of release. Neglect the inertia of the air and liquid and the effects of surface tension and assume that the temperature remains constant. (M.)

---

**Solution.** The drag on a sphere moving through a viscous fluid is a force, and so has dimensions $MLT^{-2}$, and we must obtain a combination of the other quantities so as to have these dimensions. The operative quantities will be

| | |
|---|---|
| radius $R$ | (dimensions $L$) |
| speed $V$ | (dimensions $LT^{-1}$) |
| viscosity $\mu$ | (dimensions $ML^{-1}T^{-1}$) |

and so if we assume drag $= KR^{\alpha}V^{\beta}\mu^{\gamma}$, the coefficient of proportionality $K$ being purely numerical, then equating the powers of $M$, $L$, and $T$ gives $1 = \gamma$; $1 = \alpha + \beta - \gamma$; $-2 = -\beta - \gamma$

giving $\gamma = 1$, $\beta = 1$, $\alpha = 1$ and so

$$\underline{\text{drag} = KRV\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

A small spherical air bubble at depth $y$ below the surface of a viscous liquid will have acting on it two forces only; the drag, as above, opposing motion, and the force of buoyancy urging it upwards. Neglecting inertia, surface tension, and temperature variation, we may assume these forces equal. Buoyancy varies as $gR^3\rho$ and so

$$RV\mu = K'gR^3\rho$$

or velocity $= K'gR^2\rho/\mu$.

If a small air bubble is released below the surface, and inertia is neglected, the conditions at constant temperature are

$$\text{Boyle's Law: Pressure} \times \text{Volume} = \text{Constant} \quad . \quad . \quad (2)$$

and the law of hydrostatic pressure,

$$p = g\rho y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $y$ is depth below the surface.

Since the upward velocity is $-\dfrac{dy}{dt}$, $\dfrac{dy}{dt} = -K'gR^2\rho/\mu$ i.e. since $g$, $\rho$, and $\mu$ are constant,

$$\frac{dy}{dt} \text{ varies as } -R^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From (2), $p$ varies as $R^{-3}$
and from (3) $p$ varies as $y$
so that a bubble released with constant radius at depth where the pressure is $p_1$ will have, when pressure is $p$, radius $R$ proportional to $(p_1/p)^{1/3}$

Thus from (4), $\frac{dy}{dt}$ varies as $-(p_1/p)^{2/3}$, i.e. as $-p_1^{2/3}y^{-2/3}$ . (5)

so that, by integrating, $t = -Ap_1^{-2/3}y^{5/3} + B$
$= -A^1p_1^{-2/3}p^{5/3} + B^1$

$A^1$, $B^1$ being numerical constants

There is an initial condition that time $t$ shall be measured from the point where $p = p_1$ and so

$$0 = -A^1p_1 + B^1$$

and the time to the free surface, $p = p_0$, is then given by

$$t = A^1(p_1 - p_1^{-2/3}p_0^{5/3}) \text{ as required.}$$

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**Comment.** An example of the way in which purely dimensional considerations can be invoked to obtain relationships for a simple system.

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**Problem 87**

A steady two-dimensional flow of viscous incompressible fluid takes place in an infinite channel whose walls intersect the plane of the flow in the lines $y = \pm h$, where $Oxy$ are Cartesian axes in the plane of the flow. The walls are made of porous material through which the fluid flows into the channel with a specified uniform velocity $V$ (the appropriate boundary condition on the tangential component of velocity being the usual no-slip condition). Show that it is possible to satisfy all the conditions of the problem by assuming that the stream function is of the form $Vxf(y/h)$, provided that the function $f$ satisfies the differential equation

$$f'^2 - ff'' - R^{-1}f''' = K$$

where $R = Vh/\nu$ and $K$ is a constant, and determine the corresponding pressure distribution.

It is found in practice that when the Reynolds' number $R$ is large, there are no boundary layers or other rapid changes in the velocity distribution. In this case, show that the velocity component parallel to the walls is given approximately by

$$u = \frac{\pi Vx}{2h} \cos\left(\frac{\pi y}{2h}\right) \qquad \text{(C.)}$$

---

**Solution.**

The basic equations (7.1) for two-dimensional motion are

$$\left.\begin{aligned} q_1\frac{\partial q_1}{\partial x}+q_2\frac{\partial q_1}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 q_1}{\partial x^2}+\frac{\partial^2 q_1}{\partial y^2}\right) \\ q_1\frac{\partial q_2}{\partial x}+q_2\frac{\partial q_2}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 q_2}{\partial x^2}+\frac{\partial^2 q_2}{\partial y^2}\right)\end{aligned}\right\} \quad . \quad (1)$$

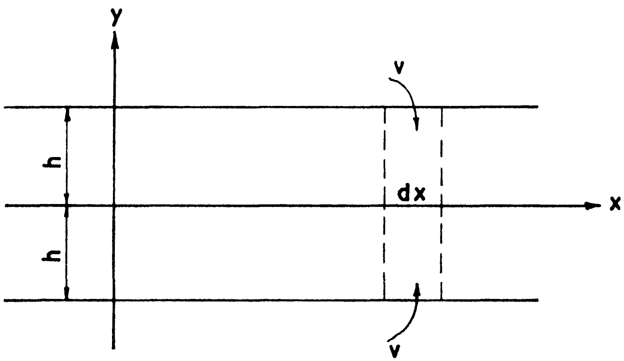


Fig. 60

The usual equation for continuity cannot be used, since fluid is flowing into the channel through the sides; the corresponding equation here, by balancing the flux into and out of a section of channel, length $\delta x$, is

$$\int_{h}^{h}\frac{\partial q_1}{\partial x}\,dy = 2V \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The boundary conditions are

$$q_1 = 0,\ y = \pm h;\ q_2 = \mp V,\ y = \pm h \quad . \quad . \quad (3)$$

Consider now the stream function $\psi = Vx\,f(y/h)$ which gives velocity components

$$\left.\begin{aligned} q_1 &= -\frac{\partial\psi}{\partial y} = -\frac{Vx}{h}f'\left(\frac{y}{h}\right) \\ q_2 &= \frac{\partial\psi}{\partial x} = V f(y/h)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

This satisfies the boundary conditions (3) if

$$f'(\pm 1) = 0, f(1) = -1, f(-1) = 1$$

It satisfies condition (2), since

$$\int_{-h}^{h}\frac{\partial q_1}{\partial x}\,dy = -\int_{-h}^{h}\frac{\partial^2\psi}{\partial x\,\partial y}\,dy = \left[-\frac{\partial\psi}{\partial x}\right]_{-h}^{h} = 2V \qquad \text{from (4) and (3)}$$

Substitution in (1) gives

$$\left.\begin{aligned}\frac{V^2x}{h^2}(f'^2 - ff'') &= -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{V^2x}{h^2}\frac{1}{R}f''' \\ \frac{V^2}{h}ff' &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{V^2}{h}\ \frac{1}{R}f''\end{aligned}\right\}$$

where $R = Vh/\nu$

i.e

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{V^2x}{h^2}\left(f'^2 - ff'' + \frac{1}{R}f'''\right) \quad . \quad . \quad (5)$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{V^2}{h}\left(ff' - \frac{1}{R}f''\right) \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Now $f$ is a function of $y$ only, so that, from (6), $\dfrac{\partial p}{\partial y}$ depends on $y$ only,

i.e. $$p = F(y) + G(x)$$

But then $\dfrac{\partial p}{\partial x} = G'(x)$, whereas, from (5), $\dfrac{\partial p}{\partial x}$ is $xH(y)$, and these are clearly incompatible. The only possibility is that $H(y) = k$, a constant, and $G'(x) = kx$.

Then from (5), $\quad f'^2 - ff'' + \dfrac{1}{R}f''' = K$ as required $\quad . \quad . \quad (7)$

The corresponding pressure distribution, integrating (6), is given by

$$-p/\rho = \frac{V^2}{h}\left(\tfrac{1}{2}f^2 - \frac{1}{R}f'\right) - G(x)/\rho \quad . \quad . \quad (8)$$

and (5), using (7), gives

$$-p/\rho = \frac{KV^2}{2h^2}x^2 - F(y)/\rho \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and combining (8) and (9) gives

$$-p/\rho = \frac{KV^2}{2h^2}x^2 + \frac{V^2}{h}\left(\tfrac{1}{2}f^2 - \frac{1}{R}f'\right) - \frac{p_0}{\rho}$$

for the pressure distribution.

The equation (7) reduces, for large $R$, to

$$f'^2 - ff'' = K \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and the boundary conditions, as above, are $f'(\pm 1) = 0$, $f(1) = -1$, $f(-1) = 1$.

A first integral of (10) satisfying these conditions is $f'^2 = K(1 - f^2)$ and, integrating again,

$$f = \sin\{\sqrt{K}(y/h + \beta)\}$$

There are many possible values of $K$, $\beta$ which will satisfy the boundary conditions. However all values except $\sqrt{K} = \pi/2$, $\beta = 2$ mean that,

within the channel, i.e. for $y/h$ between $\pm 1$, $f$ and $f'$ take both negative and positive values; and this, from (4), means that the velocities $q_1$, $q_2$ change direction within the channel. Excluding this possibility (since it certainly involves rapid changes in velocity), the only solution is

$$f = \sin\left(\frac{\pi y}{2h} + \pi\right)$$

and so the velocity component parallel to the walls, from (4), is

$$q_1 = \frac{\pi V x}{2h} \cos\frac{\pi y}{2h} \text{ as required.}$$

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**Problem 88**

A circular cylinder of radius $a$ rotates about its axis with constant angular velocity in an unbounded viscous fluid, at rest at infinity. The cylinder is porous, and fluid is sucked into the cylinder normally through its surface with constant speed $W$. Find the radial component of velocity $u(r)$ in the resulting steady motion, where $r$ is the distance from the cylinder axis.

Show that for this flow the equation $\frac{D\zeta}{Dt} = \nu\nabla^2\zeta$, satisfied by the vorticity $\zeta$ in two-dimensional flow, takes the form

$$r\frac{d^2\zeta}{dr^2} + (n+1)\frac{d\zeta}{dr} = 0 \text{ where } n = Wa/\nu$$

Deduce that the transverse component of velocity is

$$v(r) = Ar^{1-n} + Br \quad (n \neq 2)$$

where $A$, $B$ are constants. (M.)

---

**Solution.** The basic equations are

$$\mathbf{f} = \frac{D\mathbf{q}}{Dt}$$

(using $D/Dt$ to denote total rate of change, i.e. $\partial/\partial t + (\mathbf{q}.\nabla) + \Omega \wedge$, as in 7.1) and
$$\mathbf{f} = \mathbf{F} - \frac{1}{\rho}\operatorname{grad} p + \nu\nabla^2\mathbf{q} \qquad (7.1\ (1))$$

These can be transferred, in *two dimensions* only, into

$$\operatorname{curl}\mathbf{f} = \frac{D\zeta}{Dt}$$

$$\operatorname{curl}\mathbf{f} = \nu\nabla^2\zeta$$

where $\boldsymbol{\zeta} = \text{curl } \mathbf{q} = (0, 0, \zeta)$ is the vorticity, since curl $\mathbf{F} = 0$, curl $\left(\frac{1}{\rho} \text{ grad } p\right) = 0$ and curl and $D/Dt$ interchange. Thus

$$\frac{D\zeta}{Dt} = \nu\nabla^2\zeta \quad \text{. . . . . . . (1)}$$

In this problem the motion is produced purely by rotation; and so the velocity components $q_r$, $q_\theta$ depend on $r$ only. The equations are, as in 7.1.1 (*a*),

$$\text{continuity} \quad \frac{\partial q_r}{\partial r} + q_r/r = 0 \quad \text{. . . . . . . . . . (2)}$$

$$\text{vorticity} \quad \zeta = \frac{1}{2r}\frac{d}{dr}(rq_\theta) \text{ (from 1.1.5)} \quad \text{. . . . (3)}$$

and the equation (1) becomes

$$q_r \frac{d\zeta}{dr} = \nu\left(\frac{d^2\zeta}{dr^2} + \frac{1}{r}\frac{d\zeta}{dr}\right) \quad \text{. . . . . (4)}$$

The boundary conditions are $q_r = -W$, $q_\theta = \omega$, $r = a$ and $q_r = q_\theta = 0$ as $r \longrightarrow \infty$. Integrating (2) gives

$$\begin{aligned} rq_r &= \text{constant} \\ &= -Wa \text{ using the boundary conditions} \quad \text{. . (5)} \end{aligned}$$

Substituting from (5) in (4) gives

$$-\frac{Wa}{\nu r}\frac{d\zeta}{dr} = \frac{d^2\zeta}{dr^2} + \frac{1}{r}\frac{d\zeta}{dr}$$

i.e.
$$r\frac{d^2\zeta}{dr^2} + (n + 1)\frac{d\zeta}{dr} = 0 \text{ where } n = \frac{Wa}{\nu} \quad \text{. . (6)}$$

Solving (6) gives $\qquad \zeta = Ar^{-n} + B$

and substituting from (3),

$$\frac{d}{dr}(rq_\theta) = 2Ar^{-(n-1)} + 2Br$$

which, when integrated, gives, for $n \neq 2$,

$$q_\theta = A^1r^{-(n-1)} + B^1r$$

as required.

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**Problem 89**

Assuming the Navier–Stokes equations of motion for a viscous incompressible liquid, show that for slow steady motion in a plane the stream function $\psi$ satisfies the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2\psi = 0$$

Rewrite the equation in plane polars $(r, \theta)$ and determine solutions of the form $r^{n+1}f(\theta)$.

Consider the motion of the fluid in the neighbourhood of a corner formed by plane boundaries inclined at an angle to each other and show that the permissible values of $n$ are determined by the equation

$$\frac{\sin n\alpha}{n\alpha} = \pm\frac{\sin\alpha}{\alpha} \qquad \text{(L.)}$$

---

**Solution.** The Navier–Stokes equations in two dimensions for very slow motion are obtained (7.3 (*a*)) by omitting the inertia terms completely, and so are

$$\left.\begin{aligned}\frac{\partial p}{\partial x} &= \mu\nabla^2 q_1\\ \frac{\partial p}{\partial y} &= \mu\nabla^2 q_2\end{aligned}\right\} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Suppose the motion has stream function $\psi$ so that $q_1 = -\dfrac{\partial\psi}{\partial y}$, $q_2 = \dfrac{\partial\psi}{\partial x}$ and the equation of continuity is automatically satisfied.

Substituting in (1), and eliminating $p$, gives

$$\frac{\partial}{\partial y}\left\{\nabla^2\left(-\frac{\partial\psi}{\partial y}\right)\right\} = \frac{\partial}{\partial x}\left\{\nabla^2\left(\frac{\partial\psi}{\partial x}\right)\right\}$$

which means $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In plane polars, by the usual transformation (7.1.2 (*a*)) (2) becomes

$$\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right\}^2 \psi = 0 \quad . \quad . \quad . \quad . \quad (3)$$

Putting $\psi = r^{n+1}f(\theta)$ in (3) gives, after reduction,

$$r^{n-3}\{(n^2 - 1)^2 f + 2(n^2 + 1)f'' + f^{iv}\} = 0$$

so that $f$ must satisfy

$$(n^2 - 1)^2 f + 2(n^2 + 1)f'' + f^{iv} = 0 \quad . \quad . \quad . \quad (4)$$

and the general solution of this is

$$f(\theta) = A\cos(n + 1)\theta + B\sin(n + 1)\theta + C\cos(n - 1)\theta + D\sin(n - 1)\theta \quad (5)$$

Consider flow in the neighbourhood of a corner bounded by planes $\theta = 0$, $\theta = \alpha$. By the usual boundary condition (7.1.3) we must have $\frac{\partial\psi}{\partial r}$ and $\frac{1}{r}\frac{\partial\psi}{\partial\theta}$ *both* zero on *both* planes, i.e. if $\psi = r^{n+1}f(\theta)$ we require

$$f(0) = 0, f(\alpha) = 0, f'(0) = 0, f'(\alpha) = 0.$$

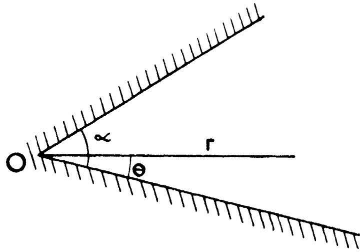


Fig. 61

Substituting in (5) gives

$$\left.\begin{aligned}
&A + C = 0\\
&A\cos(n+1)\alpha + B\sin(n+1)\alpha + C\cos(n-1)\alpha + D\sin(n-1)\alpha = 0\\
&(n+1)B + (n-1)D = 0\\
&-(n+1)A\sin(n+1)\alpha + (n+1)B\cos(n+1)\alpha\\
&\qquad -(n-1)C\sin(n-1)\alpha + (n-1)D\cos(n-1)\alpha = 0
\end{aligned}\right\}(6)$$

for which the condition that non-zero solutions exist is that

$$\begin{vmatrix}
1 & 0 & 1 & 0\\
\cos(n+1)\alpha & \sin(n+1)\alpha & \cos(n-1)\alpha & \sin(n-1)\alpha\\
0 & n+1 & 0 & n-1\\
-(n+1)\sin(n+1)\alpha & (n+1)\cos(n+1)\alpha & -(n-1)\sin(n-1)\alpha & (n-1)\cos(n-1)\alpha
\end{vmatrix} = 0$$

which reduces, after some manipulation, to

$$\sin^2 n\alpha - n^2\sin^2\alpha = 0$$

$$\frac{\sin n\alpha}{n\alpha} = \pm\frac{\sin\alpha}{\alpha} \text{ as required.}$$

*Alternatively*, the same result may be found directly from equations (6) without using the determinant, by eliminating $C$ and $D$ first and then $A$ and $B$.

**Problem 90**

Liquid of constant viscosity $\mu$ contained between concentric spheres of radii $a$ and $b$ $(b > a)$ is in slow motion due to the rotation of the spheres about a common diameter with angular velocities $\omega\mathbf{i}$ and $\omega^1\mathbf{i}$, where $\mathbf{i}$ is a fixed unit vector along this diameter. Show that the velocity $\mathbf{q}$ at any point $P$ in the liquid, and the pressure intensity $p$, are given by

$$\mathbf{q} = (Ar^{-3} + B)\mathbf{i} \times \mathbf{r}; \quad p = \text{constant}$$

where $A$ and $B$ are constants to be determined and $\mathbf{r}$ is the position vector of $P$ relative to the centre of the two spheres.

By finding the stress vector $R_r$, show that the couple which must be applied to the inner sphere to maintain its motion is

$$8\pi\mu\, a^3b^3(\omega - \omega^1)(b^3 - a^3)^{-1}\,\mathbf{i} \qquad \text{(L.)}$$

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**Solution.**

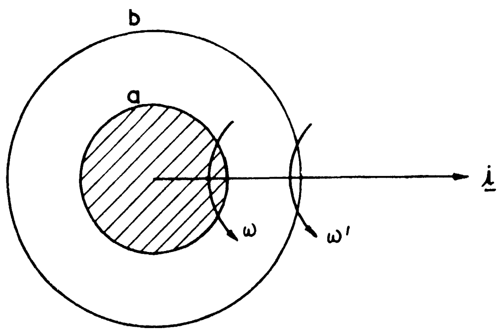


Fig. 62

The Navier–Stokes equations for very slow motion are, as in 7.3 ($a$)

$$\text{grad}\, p = \mu\nabla^2\mathbf{q} \qquad \text{. . . . . (1)}$$

Here the boundary conditions are

$$\left.\begin{aligned} \mathbf{q} &= \omega\mathbf{i} \times \mathbf{r} \quad \text{on} \quad r = a \\ \mathbf{q} &= \omega^1\mathbf{i} \times \mathbf{r} \quad \text{on} \quad r = b \end{aligned}\right\} \qquad \text{. . . . . (2)}$$

The form of the boundary conditions suggests trying, within the fluid,

$$\mathbf{q} = f(r)\,\mathbf{i} \times \mathbf{r}$$

(This gives zero velocity in the direction $\mathbf{i}$, which is reasonable), i.e. using spherical polars $(r, \theta, \tilde{\omega})$, with $\mathbf{i}$ along the initial line $\theta = 0$,

$$\mathbf{q} = \{0, 0, rf(r) \sin\theta\}$$

The equation of continuity (see 7.1.2 ($b$)) is automatically satisfied, and the equations (1) become, from 7.1.2 ($b$),

$$\frac{\partial p}{\partial r} = 0, \; \frac{\partial p}{\partial \theta} = 0 \qquad \text{. . . . . . . . (3)}$$

and

$$\frac{1}{r\sin\theta}\frac{\partial p}{\partial\tilde{\omega}} = \mu\left\{\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)q_{\tilde{\omega}} - \frac{q_{\tilde{\omega}}}{r^2\sin^2\theta}\right\}$$

which, substituting $q_{\tilde{\omega}} = rf(r)\sin\theta$ reduces to

$$\frac{1}{r\sin\theta}\frac{\partial p}{\partial\tilde{\omega}} = \mu\sin\theta(rf'' + 4f') \quad . \quad . \quad . \quad (4)$$

Thus, from (3), the pressure $p$ is independent of $r$ and $\theta$, and by the conditions of the problem, since there is symmetry about **i**, it must be independent of $\tilde{\omega}$. Hence

$$p = \text{constant}$$

and (4) becomes $$rf'' + 4f' = 0$$

for which the general solution is

$$f = Ar^{-3} + B \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Thus $\mathbf{q} = (Ar^{-3} + B)\mathbf{i} \times \mathbf{r}$ as required.
To satisfy the boundary conditions (2) we need $f(a) = \omega$, $f(b) = \omega^1$, and this gives

$$A = \frac{a^3b^3(\omega - \omega^1)}{b^3 - a^3},\; B = \frac{b^3\omega^1 - a^3\omega}{b^3 - a^3} \quad . \quad . \quad (6)$$

The stress vector on the surface of the inner sphere has only one component, that in the direction of increasing $\tilde{\omega}$, which has a moment about the axis; and, from 7.1.2 (*b*),

$$p_{r\tilde{\omega}} = \mu\left(\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial\tilde{\omega}} - \frac{q_{\tilde{\omega}}}{r} + \frac{\partial q_{\tilde{\omega}}}{\partial r}\right)$$
$$= \mu rf'\sin\theta \text{ when } q_{\tilde{\omega}} = f(r)\,r\sin\theta \quad . \quad . \quad (7)$$

The moment about the axis per unit area is then $p_{r\tilde{\omega}}\, r\sin\theta$ and so the total moment over the whole sphere is

$$\int_0^{\pi} p_{r\tilde{\omega}}\, r\sin\theta\, 2\pi r^2\sin\theta\, d\theta$$

and substituting from (7), (6), and (5) gives finally

$$\text{total moment} = -\frac{8\pi\mu(\omega - \omega^1)a^3b^3}{b^3 - a^3}$$

and this is about the axis **i**.

Thus the couple necessary to maintain the motion is

$$\frac{8\pi\mu(\omega - \omega^1)a^3b^3}{b^3 - a^3}\,\mathbf{i} \text{ as required.}$$

**Problem 91**

Give an account of Prandtl's boundary layer theory of the motion of a viscous incompressible liquid, leading to the differential equation

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U_1\frac{\partial U_1}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}$$

describing the two-dimensional steady flow along a semi-infinite flat surface, where $\nu$ is the kinematic viscosity, $U_1$ is the main stream velocity, $x$, $y$ are distances measured respectively parallel and perpendicular to the surface of the plate, and $(u, v)$ are the velocity components in the boundary layer in these directions.

Show that the steady two-dimensional flow in the boundary layer along either of the two non-parallel plane walls in a converging canal has a solution

$$u=-(c/x)\left[3\tanh^2\left\{\left(\frac{c}{2\nu}\right)^{1/2}\left(\frac{y}{x}\right)+\beta\right\}-2\right]$$

where $\tanh^2\beta = 2/3$ and $U_1 = -c/x$. (S.)

**Solution.** The boundary layer theory is developed on the assumption that:

(*a*) The effect of viscosity is apparent only in a thin layer round any boundary, the rest of the fluid being treated as non-viscous.

(*b*) The pressure distribution in the non-viscous region is known.

This may be applied to two-dimensional steady flow along a semi-infinite flat surface

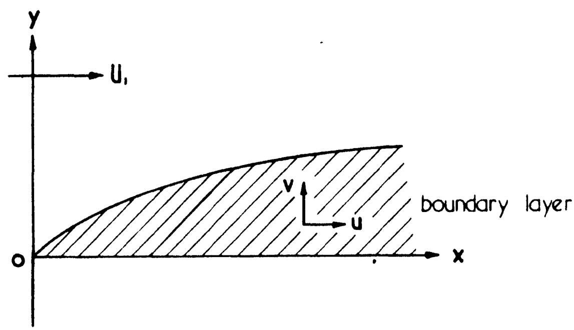


Fig. 63

Let the velocities within the boundary layer be $u$, $v$, and let the layer have thickness $\delta$ (small in comparison with any other dimensions) and the main stream be in the $x$-direction with velocity $U_1$.

Then $u$ changes from 0 at $y = 0$ (i.e. on the surface) to $U_1$ at $y = \delta$ and so $\frac{\partial u}{\partial y}$ is $O(U_1/\delta)$; $u$ itself will be $O(U_1)$, and so will $\frac{\partial u}{\partial x}$.

The equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

and so $\dfrac{\partial v}{\partial y}$ is $O(U_1)$ and, since $v=0$ on $y=0$, $v=O(U_1\delta)$ in the boundary layer, and $\dfrac{\partial v}{\partial x}$ is of the same order.

The Navier–Stokes equations (7.1) are, for steady motion in two dimensions,

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\quad . \quad . \quad . \quad (1)$$

The orders of the terms in the first equation (1) may now be written:

$$\begin{array}{llllllll} & u\dfrac{\partial u}{\partial x} & + \; v\dfrac{\partial u}{\partial y} & = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} & + \; \nu\Bigg(\dfrac{\partial^2 u}{\partial x^2} & + \; \dfrac{\partial^2 u}{\partial y^2}\Bigg) & & \\ \textit{order} & U_1^2 & U_1^2 & & \nu U_1 & \nu U_1/\delta^2 & . \; . & (2) \end{array}$$

It can be seen that $\nu\dfrac{\partial^2 u}{\partial x^2}$ may be neglected in comparison with $\nu\dfrac{\partial^2 u}{\partial y^2}$ and that $\nu\,\partial^2 u/\partial y^2$ (order $\nu U_1/\delta^2$) is of the same order as the inertia terms $u\dfrac{\partial u}{\partial x}$, $v\dfrac{\partial u}{\partial y}$ (order $U_1^2$) if

$$\delta \text{ is } O\left(\frac{\nu}{U_1}\right)^{1/2}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This gives an estimate of the boundary-layer thickness.

Applying the same reasoning to the second equation in (1) gives

$$\begin{array}{llllllll} & u\dfrac{\partial v}{\partial x} & + \; v\dfrac{\partial v}{\partial y} & = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} & + \; \nu\Bigg(\dfrac{d^2 v}{\partial x^2} & + \; \dfrac{\partial^2 v}{\partial y^2}\Bigg) & & \\ \textit{order} & U_1^2\delta & U_1^2\delta & & \nu U_1\delta & \nu U_1/\delta & . & (4) \end{array}$$

from which it follows that all terms in this equation are an order smaller than those in (2); and this applies also to the pressure terms, so that

$$\frac{\partial p}{\partial y} \text{ can be neglected in comparison with } \frac{\partial p}{\partial x}$$

This means that, for given $x$, the pressure is constant across the layer, i.e. from $y=0$ to $y=\delta$.

In the main stream, where the liquid is assumed non-viscous, Bernouilli's equation for motion under no external force gives

$$p/\rho + \tfrac{1}{2}U_1{}^2 = \text{constant}$$

and so
$$\frac{1}{\rho}\frac{\partial p}{\partial x} = -U_1\frac{dU_1}{dx}, \text{ i.e. } O(U_1{}^2) \quad . \quad . \quad . \quad (5)$$

and this relation must then hold also in the boundary layer, since the pressure is constant across it.

Thus the equation (2) reduces, taking terms of $O(U_1{}^2)$ only and using (5), to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_1\frac{dU_1}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \quad . \quad . \quad . \quad . \quad (6)$$

and the equation (4) has been shown already to reduce to

$$p = \text{constant across the boundary layer, i.e. for fixed } x$$

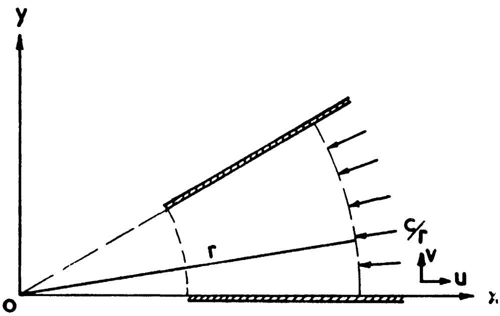


Fig. 64

When a viscous fluid flows into a convergent channel, as shown, boundary layers will build up on each wall. The main stream velocity is purely radial, and so must be that due to a two-dimensional sink, i.e. must be $c/r$ inwards at distance $r$ from $O$, the intersection of the channel walls. Considering the wall which lies along $Ox$, this means that the main stream velocity there is $-c/x$.

Thus we have to satisfy, substituting $U_1 = -c/x$ in (6)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -c^2/x^3 + \nu\frac{\partial^2 u}{\partial y^2} \quad . \quad . \quad . \quad . \quad (7)$$

continuity
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and boundary conditions

$$u = 0,\ v = 0 \text{ on } y = 0;\ u = -c/x,\ v = 0 \text{ on } y = \delta \qquad (9)$$

The form of the boundary condition (9) suggests that, within the layer,

$$u = -c/x\, f(y/x) \quad \text{where} \quad f(0) = 0 \qquad . \quad . \quad . \ (10)$$

Substituting in (8) gives

$$\frac{\partial v}{\partial y} = -(c/x^2)f - (cy/x^3)f'$$

from which, since $v = 0$, $y = 0$, $\quad v = -\dfrac{cy}{x^2}f \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$

Substituting (10) and (11) in (7) then gives, after simplification,

$$\nu f''/c = f^2 - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

It can be *verified* that the form given,

$$f(y/x) = 3\tanh^2\left\{\left(\frac{c}{2\nu}\right)^{1/2}\left(\frac{y}{x}\right) + \beta\right\} - 2$$

actually satisfies equation (12) together with the boundary condition $f(0) = 0$ if $\tanh^2\beta = 2/3$.

The equation (12) is not soluble directly by any standard method; a first integral can be obtained immediately in the form

$$\left(\frac{\nu}{2c}\right)^{1/2} f' = (\tfrac{1}{3}f^3 - f + c)^{1/2}$$

but the integral which is then required, $\int(\frac{1}{3}f^3 - f + c)^{-1/2}\,df$ is of the form which is normally an elliptic function. In this case it is actually pseudo-elliptic, but this cannot be recognised and there is no uniform method of approach (see, e.g., Hardy, *The Integration of Functions of a Single Variable*, p. 47).

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**Problem 92**

The boundary-layer equations for steady two-dimensional flow are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Prove the momentum integral equation

$$\frac{\tau_\omega}{\rho U^2} = \frac{d\delta_2}{dx} + \frac{\delta_1 + 2\delta_2}{U}\frac{dU}{dx}$$

defining the quantities introduced in this equation.

If $U$ is constant and the boundary layer starts at $x = 0$, estimate the skin friction by assuming an approximate velocity profile of the form

$$u = U\sin\tfrac{1}{2}\pi\eta, \quad \eta = y/\delta(x) \qquad \text{(M.)}$$

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**Solution.** The derivation of the boundary-layer equations is given in Problem 91 (here $U = U_1$).

Using these, the momentum integral equation follows by making some hypothesis as to the functional form for $u$, the velocity in the boundary layer, satisfying the boundary conditions, and using dynamic considerations.

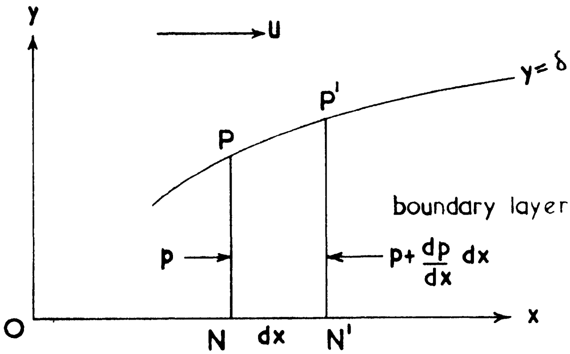


Fig. 65

Consider two adjacent points $P$, $P^1$ on the boundary layer profile $y = \delta$ ($\delta$ being a function of $x$), and let $N$, $N^1$ be the corresponding points on the $x$-axis.

The momentum in the $x$-direction of liquid within $NPP^1N^1$ is then

$$\left(\int_0^\delta \rho u \, dy\right) dx$$

The time rate of change of this must be the sum of the two quantities

(*a*) the rate of momentum flow into the area;

(*b*) the component of force in this direction across the boundary of the area.

The rate of flow of momentum into the area across $NP$ is

(*a*) $$\int_0^\delta \rho u^2 \, dy$$

and so the total rate of flow for $NP$ and $NP^1$ is

$$-\frac{\partial}{\partial x}\left(\int_0^\delta \rho u^2 \, dy\right) dx$$

The rate at which fluid mass is crossing $NP$ is $\int_0^\delta \rho u \, dy$, and so the total outflow rate, considering $NP$ and $NP^1$, is $\frac{\partial}{\partial x}\left(\int_0^\delta \rho u \, dy\right) dx$, and this

must, by continuity, equal the mass inflow rate across $PP^1$; on $PP^1$ the fluid velocity in the $x$-direction is $U$, the main stream velocity. Hence the rate of inflow of momentum in the $x$-direction across $PP^1$ is

$$U\frac{\partial}{\partial x}\left(\int_0^\delta \rho u\, dy\right) dx$$

($b$) The total component of force in the $x$-direction over boundaries $NP$, $N^1P^1$ is $-\delta \frac{dp}{dx} dx$ (it has been shown, problem 91, that pressure does not vary with $y$ across the boundary layer).

The tangential force along $NN^1$ is $p_{yx}\, dx = -\mu\left(\frac{\partial u}{\partial y}\right)_{y=0} dx$ $\left(\frac{\partial v}{\partial x}\right.$ being of smaller order, as in problem 91$\left.\right)$.

There is no tangential stress along $PP^1$ $\left(\text{since } \frac{\partial u}{\partial y} = 0,\ u = U\right)$

Thus, equating the sum of ($a$) and ($b$) to the time rate of change of momentum within the area gives

$$\frac{\partial}{\partial t}\int_0^\delta \rho u\, dy = -\frac{\partial}{\partial x}\int_0^\delta \rho u^2\, dy + U\frac{\partial}{\partial x}\int_0^\delta \rho u\, dy - \delta\frac{\partial p}{\partial x} - \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

which, in steady motion of incompressible fluid reduces to

$$\frac{\partial}{\partial x}\int_0^\delta u^2\, dy - U\frac{\partial}{\partial x}\int_0^\delta u\, dy = -\frac{\delta}{\rho}\frac{dp}{dx} - \frac{\mu}{\rho}\left(\frac{\partial u}{\partial y}\right)_{y=0} \quad . \quad (1)$$

As in problem 91, $$U\frac{dU}{dx} = -\frac{1}{\rho}\frac{dp}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and we may define three quantities:
tangential surface stress, or skin friction

$$\tau_\omega = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\left.\begin{array}{l}\text{displacement thickness} \\ \delta_1 = \dfrac{1}{U}\displaystyle\int_0^\infty (U-u)dy = \dfrac{1}{U}\int_0^\delta (U-u)dy \\ \text{momentum thickness} \\ \delta_2 = \dfrac{1}{U^2}\displaystyle\int_0^\infty u(U-u)dy = \dfrac{1}{U^2}\int_0^\delta u(U-u)dy\end{array}\right\}\begin{array}{l}\text{since } u = U \\ \text{when } y \geqslant \delta\end{array} \quad (4)$$

Substituting (2), (3), and (4) into (1) gives the momentum integral equation

$$\frac{\tau_\omega}{\rho U^2} = \frac{d\delta_2}{dx} + \frac{\delta_1 + 2\delta_2}{U}\frac{dU}{dx} \text{ as required} \quad . \quad . \quad . \quad (5)$$

If $u = U \sin \frac{1}{2}\pi\eta$, $\eta = y/\delta$, $U$ constant, $\delta = 0$ at $x = 0$, the skin friction will be given by, from (5),

$$\frac{\tau_\omega}{\rho U^2} = \frac{d\delta_2}{dx}$$

where $$\delta_2 = \delta \int_0^1 \sin \tfrac{1}{2}\pi\eta(1 - \sin \tfrac{1}{2}\pi\eta)\, d\eta = \frac{\delta(4 - \pi)}{2\pi}$$

i.e. $$\tau_\omega = \left(\frac{4 - \pi}{2\pi}\right) \rho U^2 \frac{d\delta}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Also from (3), $$\tau_\omega = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$= \frac{\mu\pi}{2\delta} U \text{ with the given expression for } u \quad (7)$$

Thus equating the two results, (6) and (7),

$$\frac{\mu\pi}{\delta} = \left(\frac{4 - \pi}{\pi}\right) \rho U \frac{d\delta}{dx}$$

from which $$\delta \frac{d\delta}{dx} = \frac{\mu\pi^2}{(4 - \pi)\rho U} = \frac{\nu\pi^2}{(4 - \pi)U}$$

By integrating, and using $\delta = 0$ when $x = 0$, and substituting in (7)

$$\delta^2 = \frac{2\nu\pi^2 x}{(4 - \pi)U}; \quad \tau_\omega = \tfrac{1}{2}\left\{\left(\frac{4 - \pi}{2}\right) \frac{\rho\mu U^3}{x}\right\}^{1/2}$$

## PROBLEMS FOR SOLUTION

1. Obtain the equation

$$\mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \frac{\partial p}{\partial z} = -P$$

satisfied by the speed $w$ of a viscous liquid flowing steadily under no external forces in straight lines parallel to the axis of $z$, $p$ being the pressure, $\mu$ the coefficient of viscosity, and $P$ a constant.

If this liquid is flowing steadily along a tube of cross-section given by

$$x^2/a^2 + y^2/b^2 = 1$$

find $w$ and show that the flux of liquid is $\dfrac{\pi a^3 b^3 P}{4\mu(a^2 + b^2)}$. (N.)

2. Viscous incompressible fluid moves two-dimensionally in planes perpendicular to the $z$-axis. At any instant the streamlines are circles about this axis. Show that the differential equation satisfied by the vorticity $\zeta$ at a distance $r$ from the $z$-axis is

$$\frac{\partial \zeta}{\partial t} = \nu\left(\frac{\partial^2 \zeta}{\partial r^2} + \frac{1}{r}\frac{\partial \zeta}{\partial r}\right)$$

where $\nu$ is the kinematic viscosity and $t$ is the time.

At $t = 0$ there is a vortex filament of strength $K$ positioned along the $z$-axis, and elsewhere the vorticity vanishes. Find the velocity of the fluid at distance $r$ from the axis at time $t$ and show that if a circle, with its centre on the axis, is drawn so as to enclose a constant amount of vorticity its radius must increase steadily with time. Show also that the circulation inside the circle $r = R$ in the plane $z = 0$ is reduced to half its initial value in time $R^2/4\nu \log 2$. (S.)

3. Give an account of Prandtl's boundary layer theory of the motion of a viscous incompressible liquid.

Obtain the equation $u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} - \nu\dfrac{\partial^2 u}{\partial y^2} = U\dfrac{\partial U}{\partial x}$ for the steady flow along a boundary $y = 0$, the main stream velocity outside the layer being $U$ parallel to the boundary. Show that if $U$ is constant, then

$$\frac{d}{dx}\int_0^\delta (u^2 - uU)dy + \nu\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$$

where $\delta(x)$ is the thickness of the layer at coordinate $x$. If the liquid streams past the flat plate $y = 0$, $0 \leqslant x \leqslant l$, in steady flow with $U$ constant, find $\delta(x)$ given that $u = (Uy/\delta)(2 - y/\delta)$ and show that the drag on both sides of the plate is

$$\left(\frac{32 l\nu\rho^2 U^3}{15}\right)^{1/2} \qquad \text{(L.)}$$

4. Incompressible liquid, whose coefficient of viscosity is $\mu$, occupies the region between two infinite cylinders, one containing the other, their generators being parallel to $Oz$. The outer cylinder is fixed while the inner has a uniform velocity $V$ in the direction $Oz$. Assuming that the pressure is constant, show that in the absence of external forces the equations of motion are satisfied if the velocity at the point $(x, y, z)$ is $(0, 0, w)$ where $w$ is independent of $z$ and $t$ and satisfies Laplace's equation.

Show that the drag per unit length on the outer cylinder is $\mu\displaystyle\int\frac{\partial w}{dn}ds$ taken round the boundary of its cross-section, $\dfrac{\partial w}{\partial n}$ being the normal derivative of $w$. By using the transformation $x = c\cosh\xi\cos\eta$, $y = c\sinh\xi\sin\eta$, or otherwise show that when the cylinders are $x^2 + 4y^2 = a^2$, $5x^2 + 8y^2 = 10a^2$ the drag is

$$4\pi\mu V/\log\left\{\frac{13 + 4\sqrt{10}}{9}\right\} \qquad \text{(L.)}$$

5. Find the velocity distribution in the steady flow of viscous incompressible fluid along an infinitely long circular pipe due to an applied pressure gradient.

A large cylindrical tank, the horizontal circular end of which has radius $a$, is filled with water to a depth $h$, which is of the same order of magnitude as $a$, the free surface being open to the atmosphere. The water drains from the bottom of the tank through a long and narrow vertical pipe, of uniform circular section, which is open to the atmosphere at its lower end, and, simultaneously, the level of water in the tank is maintained by an external supply. Show that, if the external supply is cut off, the tank will empty itself in a time

$$\frac{8a^2\nu l}{b^4 g}\log(1 + h/l)$$

approximately, where $l$ and $b$ are the length and radius of the pipe and $\nu$ is the kinematic viscosity of the water.

Discuss any approximations you make, and explain how their validity is related to the restrictions on the various lengths occurring in the problem. (C.)

6. Show that with a suitable choice of units the Navier–Stokes equations can be expressed in the form

$$\dot{u} + uu_x + vu_y + wu_z = -p_x + \epsilon^2\nabla^2 u \text{ etc.}$$

where $\epsilon^{-2}$ is the Reynolds' number of the flow.

Hence obtain Prandtl's boundary-layer equations, and construct an approximate solution for a semi-infinite flat plate placed in and parallel to a uniform stream. (S.)

7. A viscous incompressible liquid flows slowly with axial symmetry about a fixed straight line. The coordinates of any point $P$ in the liquid are denoted by $R$, the perpendicular distance of $P$ from the fixed line, and $x$, the distance of the foot of this perpendicular from a fixed point $O$ on the line. Show that the components of vorticity $(\xi, \eta, \zeta)$ and the stream function $\psi$ satisfy the equations $\nabla^2\xi = \nabla^2\eta = \nabla^2\zeta = 0$ and

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial R^2} - \frac{1}{R}\frac{\partial}{\partial R}\right\}^2 \psi = 0$$

By obtaining a suitable solution of this equation, show that for the slow motion of a viscous fluid past a stationary sphere, of radius $a$,

$$\psi = -\tfrac{1}{2}UR^2\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)$$

where $r$ is the distance $OP$, and $U$ the main stream velocity. (S.)

8. A viscous incompressible liquid streams with velocity $U\mathbf{i}$ past a body with axis of symmetry $Ox$. Deduce Oseen's equations in the form

$$\left(\nabla^2 - 2K\frac{\partial}{\partial x}\right)\mathbf{q}' = 0, \quad p = \rho U\frac{\partial\phi}{\partial x}, \quad 2K = U/\nu$$

where $\mathbf{q}$, the fluid velocity, is $\mathbf{q} = U\mathbf{i} - \nabla\phi + \mathbf{q}'$

Verify that these equations possess a solution in the form

$$\mathbf{q}' = \frac{1}{2K}\nabla\chi - \chi\mathbf{i}$$

where $\chi = \chi(x, \tilde{\omega})$ and $(\nabla^2 - K^2)e^{-Kx}\chi = 0$

Find also the vorticity at any point. (E.)

9. Viscous fluid fills the space between rigid boundaries which lie in the planes $y = 0$, $y = h$ and which are made to move parallel to the $x$-axis with velocities $U\cos nt$, $-U\cos nt$ respectively. Assuming the mean pressure to be constant, prove that the velocity of the fluid at $(x, y)$ is

$$Ue^{-Ky}\cos(nt - Ky) + a\{e^{Ky}\cos(nt + Ky) - e^{-Ky}\cos(nt - Ky)\}$$
$$- b\{e^{Ky}\sin(nt + Ky) - e^{-Ky}\sin(nt - Ky)\}$$

where

$$a = U\frac{(e^{-Kh}\cosh Kh - \cos Kh\sinh Kh - \cos^2 Kh)}{2(\cosh^2 Kh - \cos^2 Kh)}$$
$$b = \tfrac{1}{2}U\sin Kh/(\cosh Kh - \cos Kh)$$
$$K^2 = n/2\nu$$

(O.)

10. Explain the nature of the flow of incompressible viscous fluid past an obstacle when the Reynolds' number is very small, and show that in this case the velocity vector $\mathbf{v}$ is approximately a solution of the equation

$$\text{curl}^3\,\mathbf{v} = 0$$

Using spherical polar coordinates $(r, \theta, \tilde{\omega})$ show that $\mathbf{v} = f(r) \sin\theta\, \mathbf{K}$ will be a solution when

$$\frac{1}{r}\frac{d^2(rf)}{dr^2} - \frac{2f}{r^2} = 0$$

$\mathbf{K}$ being a unit vector in the direction of $\tilde{\omega}$ increasing.

Hence find the solution appropriate to the slow steady rotation of a sphere in an unbounded fluid which is otherwise at rest, and show that the couple necessary to maintain the rotation is

$$G = 8\pi\mu\omega a^3$$

where $\mu$ is the coefficient of viscosity, $\omega$ the angular velocity and $a$ the radius.

(L.)